Cheeger Inequalities for Vertex Expansion, Directed Graphs and Hypergraphs via Reweighted Eigenvalue

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Introduction

Let $G = (V, E)$ be a graph. Its *conductance* is

$$\phi(G) := \min_{\emptyset \neq S \subset V} \phi(S) = \min_{\emptyset \neq S \subset V} \frac{\#\text{ edges crossing } S}{\min(\text{total degree in } S, \text{ total degree in } S^c)}.$$

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$ be the eigenvalues of the normalized Laplacian $L := I - D^{-1/2}AD^{-1/2}$. The classical Cheeger's Inequality states that:

$$\frac{\lambda_2}{2} \leq \phi \leq \sqrt{2\lambda_2}.$$

We develop the theory of *reweighted eigenvalues* to extend this to vertex expansion, directed graphs, and hypergraphs.

Motivation: Fastest Mixing Time

Boyd, Diaconis and Xiao [1] studied the problem of fastest mixing time. Consider this example:

Figure 1. The graph is two copies of K_n connected by a perfect matching. Its mixing time is $\Theta(n)$. We can improve the mixing time to $\Theta(1)$ if the perfect matching edge weights are increased to n.

Objective is to find a π-reversible Markov chain P supported on E, that maximizes $\lambda_2(P)$:

$$\begin{aligned} \lambda_2^*(G) \ := \ \max_{P \geq 0} \ & \lambda_2(P) \\ \text{subject to} \ & P(u,v) = P(v,u) = 0 && \forall uv \notin E \\ & \sum_{v \in V} P(u,v) = 1 && \forall u \in V \\ & \pi(u)P(u,v) = \pi(v)P(v,u) && \forall uv \in E. \end{aligned}$$

Cheeger Inequality for Vertex Expansion [9, 6]

Let $\pi : V \to \mathbb{R}^+$ be vertex weights and Δ the max degree of the graph. Define *vertex expansion*:

$$\psi(G) := \min_{\emptyset \neq S \subset V} \psi(S) = \min_{\emptyset \neq S \subset V} \frac{\pi(N(S))}{\min(\pi(S), \pi(S^c))}.$$

Then,

$$\lambda_2^* \lesssim \psi \lesssim \sqrt{\lambda_2^* \cdot \log \Delta}.$$

Moreover, we can find in polynomial time a set $S \subset V$ such that $\psi(S) \lesssim \sqrt{\lambda_2^* \cdot \log \Delta}$.

The proof consists of rounding the following dual program (due to Roch [10]):

$$\begin{aligned} \gamma^{(n)}(G) := \min_{\substack{f: V \to \mathbb{R}^n \\ g: V \to \mathbb{R}_{\geq 0}}} \ & \sum_{v \in V} \pi(v) g(v) \\ \text{subject to} \ & \sum_{v \in V} \pi(v) \|f(v)\|^2 = 1 \\ & \sum_{v \in V} \pi(v) f(v) = \vec{0} \\ & g(u) + g(v) \geq \|f(u) - f(v)\|^2 && \forall uv \in E. \end{aligned}$$

There are three steps:

1. Gaussian projection to $\gamma^{(1)}$ where f is now a one-dimensional embedding ($\log \Delta$ loss)
2. Going from the "ℓ_2^2 program" to the "ℓ_1 program" (square-root loss)
3. Threshold rounding à la classical Cheeger

Past Spectral Formulations for Directed Graphs

It is **tricky** to develop a spectral theory for directed graphs. For one, the Laplacian is not symmetric. Past attempts include:

- Cheeger constant for directed graphs by Fill [3] and Chung [2];
- Cheeger inequality for nonlinear Laplacian by Yoshida [12]; etc.

Expansion Quantities on Directed Graphs

We are interested in the following quantities.

- Directed edge conductance (w-weighted edges):

$$\vec{\phi}(G) := \min_{\emptyset \neq S \subset V} \vec{\phi}(S) = \min_{\emptyset \neq S \subset V} \frac{\min(\text{weights of edges leaving } S, \text{ weights of edges leaving } S^c)}{\min(\text{total weighted degree in } S, \text{ total weighted degree in } S^c)}.$$

- Directed vertex expansion (π-weighted vertices):

$$\vec{\psi}(G) := \min_{\emptyset \neq S \subset V} \vec{\psi}(S) = \min_{\emptyset \neq S \subset V} \frac{\min(\pi(\text{out-neighbors of } S), \pi(\text{out-neighbors of } S^c))}{\min(\pi(S), \pi(S^c))}.$$

Main Idea: Eulerian Reweighting

For directed graphs, we consider vertex/edge-capacitated *Eulerian* reweightings, because:

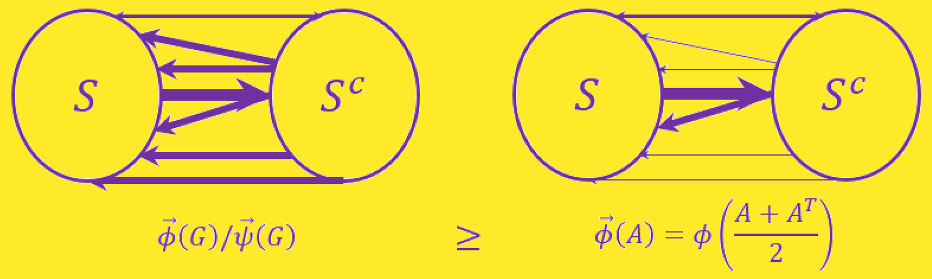

Figure 2. Edge weight from S to S^c (and from S^c to S) in any edge-capacitated Eulerian reweighting A is at most that from S to S^c in the original graph.

The goal is to search for the "best" Eulerian reweighting A that certifies conductance/vertex expansion of graph G. To this end, we maximize λ_2 of the underlying undirected graph $\frac{A+A^T}{2}$.

Cheeger Inequalities for Directed Graphs [7]

Reweighted eigenvalue objective for directed edge conductance is:

$$\begin{aligned} \vec{\lambda}_2^{e*}(G) \ := \ \max_{A \geq 0} \ & \lambda_2\left(D^{-\frac{1}{2}}\left(D_A - \frac{A+A^T}{2}\right)D^{-\frac{1}{2}}\right) \\ \text{subject to} \ & A(u,v) = 0 && \forall uv \notin E \\ & \sum_{v \in V} A(u,v) = \sum_{v \in V} A(v,u) && \forall u \in V \\ & A(u,v) \leq w(uv) && \forall uv \in E \end{aligned}$$

Then,

$$\vec{\lambda}_2^{e*} \lesssim \vec{\phi} \lesssim \sqrt{\vec{\lambda}_2^{e*} \cdot \log(1/\vec{\lambda}_2^{e*})}.$$

Rounding algorithm follows the same outline, with some tweaks. For $\vec{\lambda}_2^{v*}$ similarly defined,

$$\vec{\lambda}_2^{v*} \lesssim \vec{\psi} \lesssim \sqrt{\vec{\lambda}_2^{v*} \cdot \log(\Delta/\vec{\lambda}_2^{v*})}.$$

Hypergraphs

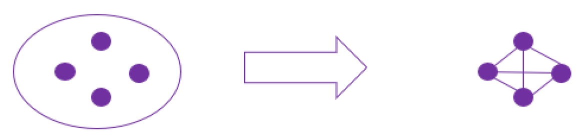

Figure 3. For hypergraphs, we consider reweighting their *clique-graphs*.

Generalizations

Some extensions of Cheeger's inequality (undirected, edge) have close analogies in the new settings, but not others.

	undirected, vertex	directed, edge	directed, vertex	hypergraphs
"bipartite" Cheeger [11]	✓	✗	✗	✗
"higher-order" Cheeger [8]	✓	✗	✗	✓
"improved" Cheeger [5]	✓	✓	✓	✓

Applications

Certifying expanders in directed graphs

We see that $\vec{\phi}(G)$ is constant iff $\vec{\lambda}_2^{e*}(G)$ is constant.

Fastest mixing time of non-reversible Markov chains

By combining Cheeger inequality for $\vec{\psi}(G)$ and some mixing time argument, we arrive at:

$$\frac{1}{\vec{\psi}(G)} \cdot \frac{1}{\log(1/\pi_{\min})} \lesssim \tau^*(G) \lesssim \frac{1}{\vec{\psi}(G)^2} \cdot \log \frac{\Delta}{\vec{\psi}(G)} \cdot \log \frac{1}{\pi_{\min}}.$$

Future Directions

1. Determine if the Cheeger inequalities for directed graphs are tight.
2. Find practical applications for this spectral algorithm.
3. Design a fast (e.g. almost-linear time) algorithm to compute reweighted eigenvalues or Cheeger cuts for directed graphs.
4. Formulate generalizations of Cheeger's inequality (bipartite, higher-order) for directed graphs.

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