# Reweighted Eigenvalues

## A New Approach to Spectral Theory beyond Undirected Graphs


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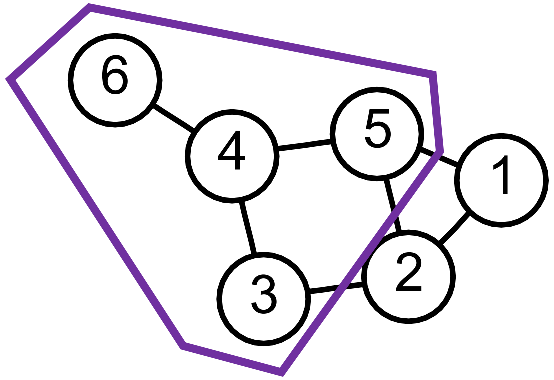
# What is Spectral Graph Theory?

- Graphs




Vertex Set

Edge Set

$$G = (V, E)$$

- Laplacian Matrix

(Unnormalized)
Laplacian Matrix

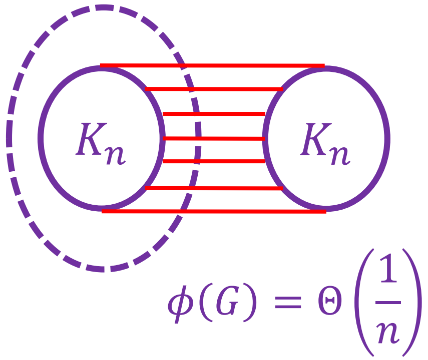
$$L' = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} = D - A'$$

Degree Matrix

(Unnormalized)
Adjacency Matrix

**Spectral graph theory studies graphs through eigenvalues and eigenvectors of associated matrices**

# Cheeger's Inequality




- Conductance of cut $S \subseteq V$ and of graph $G$

$$\phi(S) := \frac{\#\text{edges crossing } S}{\text{volume (total degree) of } S} \quad \text{and} \quad \phi(G) := \min_{S: vol(S) \leq \frac{vol(V)}{2}} \phi(S)$$

- Eigenvalues of *normalized* Laplacian $L := D^{-\frac{1}{2}} L' D^{-\frac{1}{2}}$
  - $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$

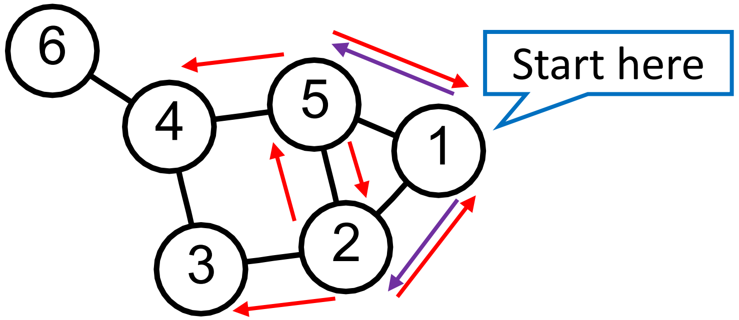
Theorem (Cheeger's Inequality) [Cheeger '70, Alon, Milman '85, Alon '86]

$$\frac{\phi(G)^2}{2} \leq \lambda_2(G) \leq 2\phi(G)$$

- Sanity check: $G$ disconnected $\Leftrightarrow \phi(G) = 0 \Leftrightarrow \lambda_2(G) = 0$

# Conductance, Eigenvalue, and Mixing Time

- Random Walk on $G$: from $u$, go to uniformly random neighbour

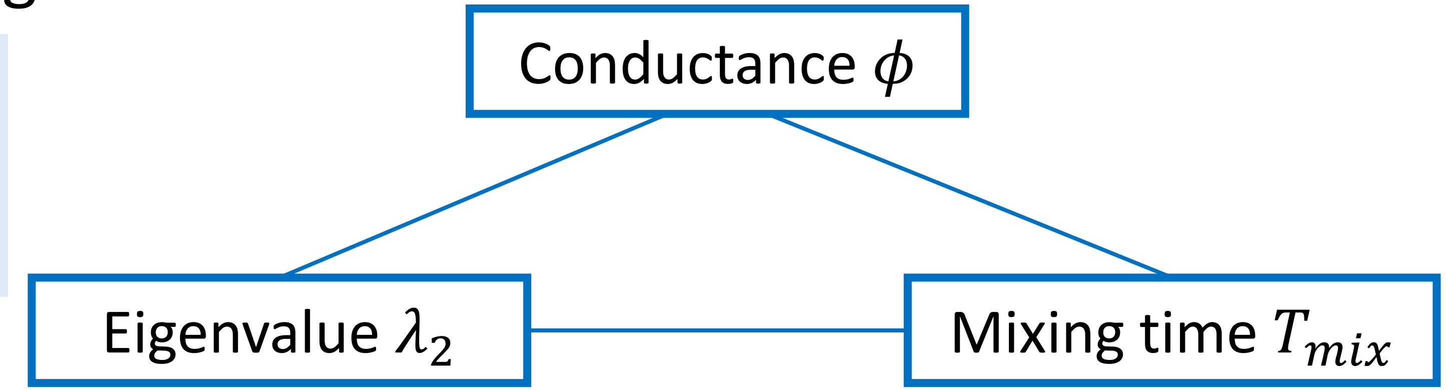



| $t$ | $p_1^{(t)}$ | $p_2^{(t)}$ | $p_3^{(t)}$ | $p_4^{(t)}$ | $p_5^{(t)}$ | $p_6^{(t)}$ |
|---|---|---|---|---|---|---|
| **0** | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| **1** | $0$ | $1/2$ | $0$ | $0$ | $1/2$ | $0$ |
| **2** | $1/3$ | $1/6$ | $1/6$ | $1/6$ | $1/6$ | $0$ |

- Mixing time $T_{mix}$: time needed to get $(1/e)$-close to stationary distribution from *any* starting distribution

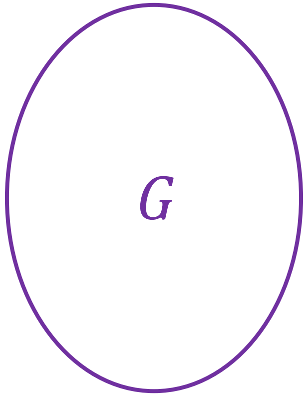
<u>Theorem</u> ($\lambda_2$ and Mixing Time)

$$\frac{1}{\lambda_2} \lesssim T_{mix} \lesssim \frac{\log n}{\lambda_2}$$



# Sweep-Cut Algorithm

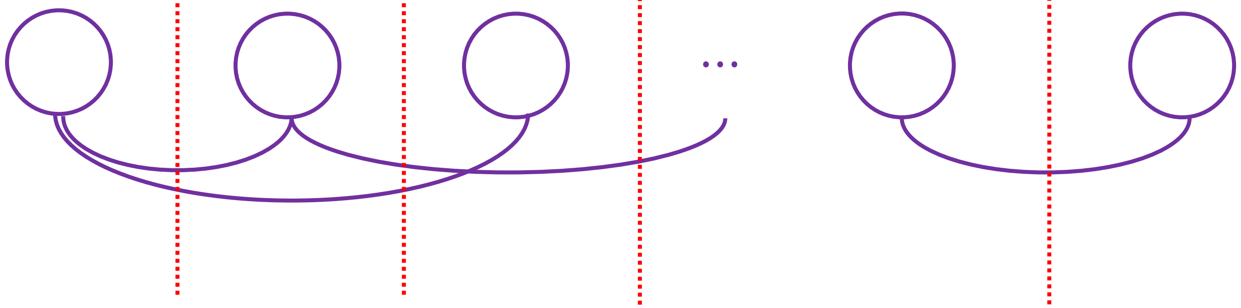
Input: Graph $G = (V, E)$




Output: cut $S \subseteq V$

Sort by second eigenvector $f$

$f(u_1) \leq f(u_2) \leq f(u_3) \leq \cdots \leq f(u_{n-1}) \leq f(u)$




$S_i :=$ $\{u_1, \dots, u_i\}$ or $\{u_{i+1}, \dots, u_n\}$, find min conductance

- Analysis:
  - Output $S \subseteq V$ satisfies $vol(S) \leq \frac{vol(V)}{2}$ and $\phi(S) \leq 2\sqrt{\phi(G)}$
  - Runtime is $O(n \log n + m)$, near linear in input size!

By proof of Cheeger's inequality

# Applications of Cheeger's Inequality

- Cheeger cut can be found in near-linear time
  - Computing the optimal cut is NP-hard

- Applications:
  - Graph partitioning (e.g. image segmentation)
  - **Expander certification**
  - Laplacian solvers
  - Divide and conquer algorithms
  - Mixing time analysis
  - etc.

$\phi(G) = \Theta(1)$ iff $\lambda_2(G) = \Theta(1)$

# “Cheeger generalizations”

| Alias | References | Informal Statement |
|---|---|---|
| “Bipartite” | [Trevisan ‘09] | $\lambda_n$ is close to 2 $\Leftrightarrow$ $G$ has an (almost) bipartite sparse cut |
| “Higher-Order” | [Lee, Oveis Gharan, Trevisan ‘12]<br>[Louis, Raghavendra, Tetali, Vempala ’12] | $\lambda_k$ is small $\Leftrightarrow$ $G$ has $O(k)$ disjoint sparse cuts |
| “Improved” | [Kwok, Lau, Lee, Oveis Gharan, Trevisan ’13] | $\lambda_{O(1)}$ is large $\Rightarrow$ spectral partitioning algorithm output $S$ has conductance $O(\phi(G))$ instead of $O(\sqrt{\phi(G)})$ |

**These generalizations enrich spectral graph theory for undirected graphs using Laplacian eigenvalues**

# Other Measures of Isoperimetry

- Edge Expansion

$$\phi'(S) \coloneqq \frac{\#\text{edges crossing } S}{|\mathrm{S}|} \quad \text{and} \quad \phi'(G) \coloneqq \min_{S:|S|\leq\frac{|V|}{2}} \phi'(S)$$

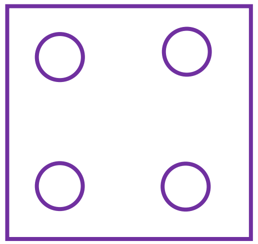
- Vertex Expansion

$$\psi(S) \coloneqq \frac{\#\text{neighbours of } S}{|\mathrm{S}|} \quad \text{and} \quad \psi(G) \coloneqq \min_{S:|S|\leq\frac{|V|}{2}} \psi(S)$$

# Other Graph Models

- Directed Graphs

- Hypergraphs



- Directed Hypergraphs
  - Common generalization of the above

# Research Gap

- There have been many efforts to derive a spectral theory for these more general settings
- **Nevertheless, these existing theories fail to fully capture the rich applications of the basic spectral theory…**

# Reweighted Eigenvalues

- We propose “reweighted eigenvalues” as a new, unifying spectral theory on these general settings
- **It reduces these general settings to the basic setting of conductance on undirected graphs**
- The result is a rich spectral theory that captures:
  - Cheeger’s inequalities for these settings
  - Direct analogues of “Cheeger generalizations” for these settings
  - Spectral certification of directed expander graphs
  - and so on…

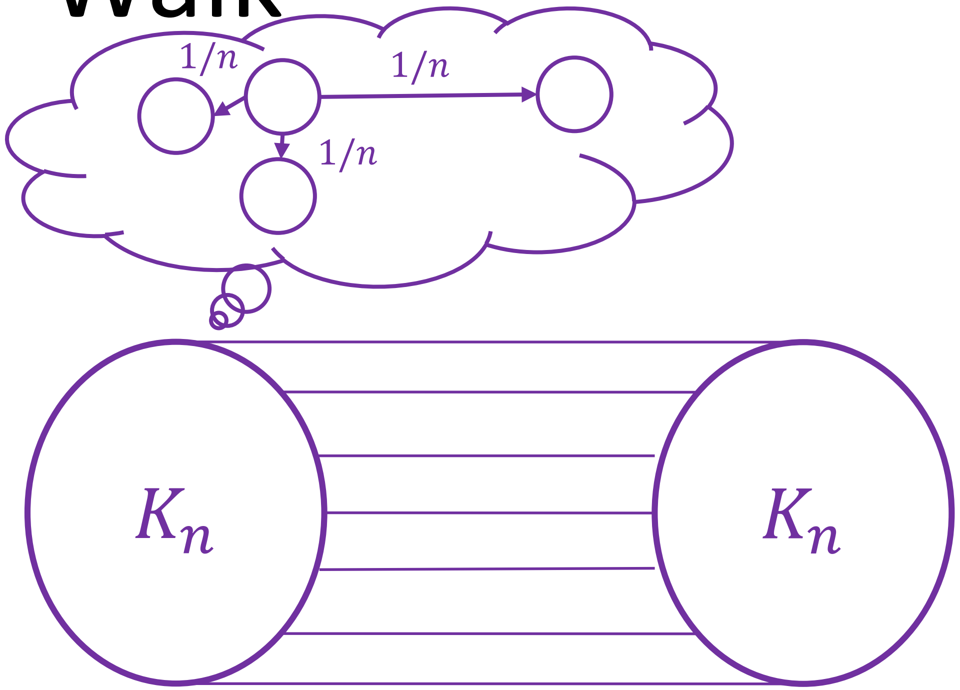
# Related Publications

- Tsz Chiu Kwok, Lap Chi Lau, Kam Chuen Tung. ***Cheeger inequalities for vertex expansion and reweighted eigenvalues***
  - Accepted to FOCS 2022
- Lap Chi Lau, Kam Chuen Tung, Robert Wang. ***Cheeger inequalities for directed graphs and hypergraphs using reweighted eigenvalues***
  - Accepted to STOC 2023
- Lap Chi Lau, Kam Chuen Tung, Robert Wang. ***Fast algorithms for directed graph partitioning using flows and reweighted eigenvalues***
  - Accepted to SODA 2024

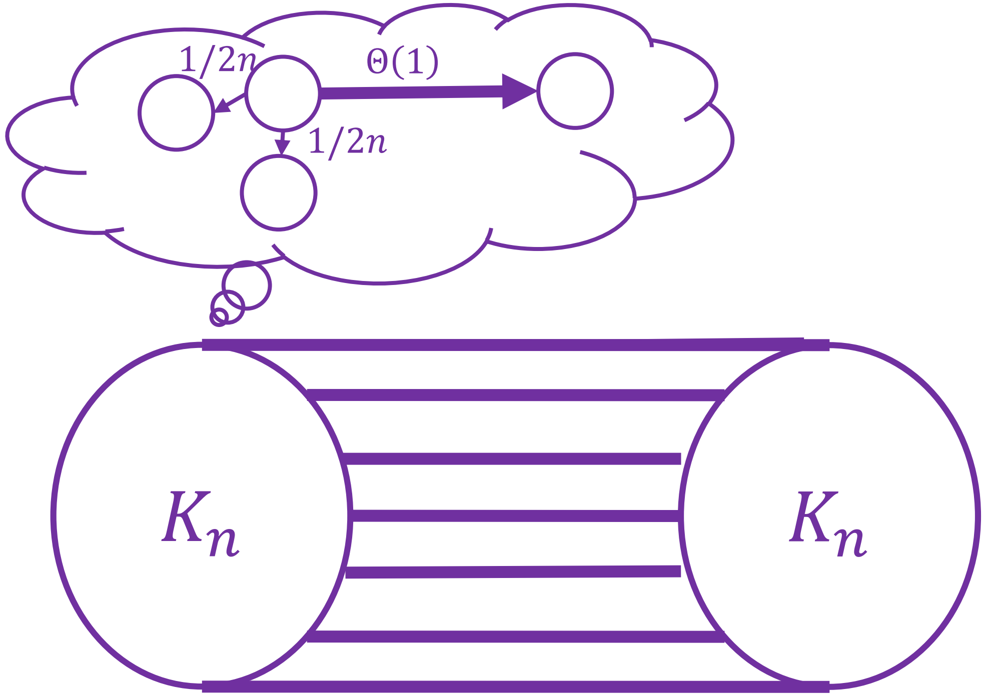
# Cheeger Inequalities for Vertex Expansion using Reweighted Eigenvalues

Joint work with Tsz Chiu Kwok and Lap Chi Lau

# Motivation: “Improving” Graph Random Walk




Mixing time is $\Theta(n)$




Mixing time is $\Theta(1)$

# Fastest Mixing Markov Chain (FMMC)

- Given a target stationary distribution $\pi$ on $V$
- Goal: Find a *reversible* Markov chain supported on the graph edges $E$, s.t. the mixing time to $\pi$ is small
- Equivalently:
  - Assign weights $A(u, v) \geq 0$ to each edge $uv \in E$ or $u = v$
  - Induced Markov chain $P(u, v) = A(u, v) / \deg_A(u)$
  - Require: $\deg_A(u) = \pi(u)$ for all $u \in V$

- Fastest mixing time $T^*_{mix}$ defined as $\min_P T_{mix}(P)$

# SDP Formulation of FMMC [Boyd, Diaconis, Xiao '04]

Since $T_{mix}(P) \approx \lambda_2(I-P)^{-1}$

- Maximizing $\lambda_2(I-P)$ as a proxy for minimizing $T_{mix}(P)$

$$
\begin{aligned}
\lambda_2^*(G) \;:=\; \max_{P \geq 0} \quad & 1-\alpha_2(P) & \\
\text{subject to} \quad & P(u,v) = P(v,u) = 0 & \forall uv \notin E \\
& \sum_{v \in V} P(u,v) = 1 & \forall u \in V \\
& \pi(u)P(u,v) = \pi(v)P(v,u) & \forall uv \in E,
\end{aligned}
$$

- $\lambda_2^* \lesssim \psi$ [Roch '05]
  - Small vertex expansion $\Rightarrow$ torpid fastest mixing
- $\psi^2/\log n \lesssim \lambda_2^* \lesssim \psi$ [Olesker-Taylor, Zanetti '22]
  - First Cheeger-like relation between vertex expansion and $\lambda_2^*$
  - Proven for uniform distribution

# Our Results

- Optimal Cheeger's inequality for vertex expansion

<u>Theorem</u> (Cheeger's Inequality for Vertex Expansion) [Kwok, Lau, **T.** '22]

$$\frac{\psi(G)^2}{\log \Delta} \lesssim \lambda_2^*(G) \leq 2\psi(G)$$

- Analogue of "Cheeger generalizations" using $\lambda_k^*(G)$
- 0/1 polytope with torpid fastest mixing

# Review: Proof Flow of Cheeger's Inequality

$$\lambda_2 = \min_{f \perp deg} \frac{\sum_{uv\in E}\left(f(u)-f(v)\right)^2}{\sum_{v\in V} \deg(v)\, f(v)^2}$$

Rayleigh quotient for $\lambda_2$

$\Downarrow$ (1) $\ell_2^2$ to $\ell_1$ step

$$\xi = \min_{M_{deg}(h)=0} \frac{\sum_{uv\in E}|h(u)-h(v)|}{\sum_{v\in V} \deg(v)\,|h(v)|}$$

Key technique: Cauchy-Schwarz Inequality

$deg$-weighted median

$\Downarrow$ (2) threshold rounding

$$\phi = \min_{\substack{M_{deg}(x)=0\\ x(u)\in\{0,1\}}} \frac{\sum_{uv\in E}|x(u)-x(v)|}{\sum_{v\in V} \deg(v)\,|x(v)|}$$

$x(u)$ depends on if $h(u) > t$ for random threshold $t \in \mathbb{R}$

$\Downarrow$

$$S = \{u \in V : x(u) = 1\}$$

$$\phi(S) \overset{(2)}{\leq} \xi \overset{(1)}{\leq} \sqrt{2\lambda_2}$$

# Abstracting the Proof Flow

$$\lambda_2 = \min_f \frac{\ell_2^2 \text{ energy of } f}{\ell_2^2 \text{ mass of } f}$$

$\Downarrow$ (1) $\ell_2^2$ to $\ell_1$ step

$$\xi = \min_h \frac{\ell_1 \text{ energy of } h}{\ell_1 \text{ mass of } h}$$

$\Downarrow$ (2) threshold rounding

$$\phi = \min_{S_t} \frac{\text{edge boundary of } S_t}{\text{volume of } S_t}$$

$\Downarrow$

$S_t$ with minimum conductance

$$\phi(S) \overset{(2)}{\leq} \xi \overset{(1)}{\leq} \sqrt{2\lambda_2}$$

# Dual program [Roch '05]

$$\lambda_2^* = \max_{P\ rwt.}\ \min_{f \perp \pi} \frac{\sum_{uv\in E} \pi(u) P(u,v) \big(f(u) - f(v)\big)^2}{\sum_{v\in V} \pi(v)\, f(v)^2}$$

$\Downarrow$ Von Neumann minimax (require $F: V \to \mathbb{R}^n$)

$$\gamma^{(n)} = \min_{F_i \perp \pi}\ \max_{P\ rwt.} \frac{\sum_{uv\in E} \pi(u) P(u,v) \|F(u) - F(v)\|^2}{\sum_{v\in V} \pi(v)\, \|F(v)\|^2}$$

$\Downarrow$ LP dual of inner max program

$$\gamma^{(n)} = \min_{F_i \perp \pi} \frac{\min_{g\geq 0} \sum_{v\in V} \pi(u) g(u)}{\sum_{v\in V} \pi(u) \|F(u)\|^2}$$

$$\text{s.t. } g(u) + g(v) \geq \|F(u) - F(v)\|^2\ \forall uv \in E$$

$\ell_2^2$ objective of $F$

$\ell_2^2$ mass of $F$

# Proof Flow of KLT22

$$\gamma^{(n)} = \min_F \frac{\ell_2^2 \text{ objective of } F}{\ell_2^2 \text{ mass of } F} \quad \Rightarrow \quad \gamma^{(1)} = \min_f \frac{\ell_2^2 \text{ objective of } f}{\ell_2^2 \text{ mass of } f}$$

Gaussian Projection

Strong duality [Roch '05] ⇧

$$\lambda_2^* = \text{reweighted eigenvalue}$$

⇩ (1) $\ell_2^2$ to $\ell_1$ step

$$\xi = \min_h \frac{\ell_1 \text{ objective of } h}{\ell_1 \text{ mass of } h}$$

⇩ (2) threshold rounding

$$\psi = \min_{S_t} \frac{\text{vertex boundary of } S_t}{\pi\text{−weight of } S_t}$$

⇩

$$S_t \text{ with minimum vertex expn.}$$

$$\psi(S) \overset{(2)}{\leq} \xi \overset{(1)}{\leq} \sqrt{\gamma^{(1)}} \lesssim \sqrt{\gamma^{(n)} \cdot \log \Delta}$$

Polynomial time!

# Higher Reweighted Eigenvalues

- $\lambda_k^*$ defined as $\max_P \lambda_k(I-P)$, not convex
- Idea: Consider $\sigma_k^*(I-P) = \max_P \sum_{i\le k} \lambda_i(I-P)$

| Alias | References | Informal Statement |
|---|---|---|
| "Bipartite" | [Trevisan '09] | $\lambda_n$ is close to 2 $\Leftrightarrow$ $G$ has an (almost) bipartite sparse cut |
| "Higher-Order" | [Lee, Oveis Gharan, Trevisan '12]<br>[Louis, Raghavendra, Tetali, Vempala '12] | $\lambda_k$ is small $\Leftrightarrow$ $G$ has $O(k)$ disjoint sparse cuts |
| "Improved" | [Kwok, Lau, Lee, Oveis Gharan, Trevisan '13] | $\lambda_{O(1)}$ is large $\Rightarrow$ spectral partitioning algorithm output $S$ has conductance $O(\phi(G))$ instead of $O(\sqrt{\phi(G)})$ |
| "Bipartite" | [Kwok, Lau, **T.** '22] | $\lambda_n^*$ is close to 2 $\Leftrightarrow$ $G$ has an (almost) bipartite small-$\psi$ cut |
| "Higher-Order" | [Kwok, Lau, **T.** '22] | $\lambda_k^*$ is small $\Leftrightarrow$ $G$ has $O(k)$ disjoint small-$\psi$ cuts |
| "Improved" | [Kwok, Lau, **T.** '22] | $\lambda_{O(1)}^*$ is large $\Rightarrow$ spectral partitioning algorithm output $S$ has $\psi(S) \le O(\psi(G)\cdot\log\Delta)$ instead of $O(\sqrt{\psi(G)\cdot\log\Delta})$ |

# 0/1 Polytope Conjecture

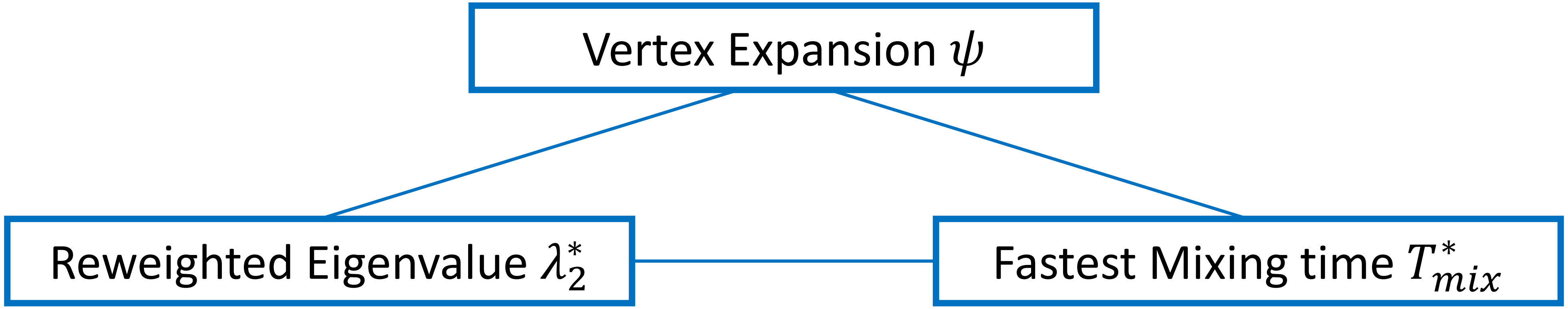
- A 0/1 polytope is a polytope whose vertex set is a subset of $\{0,1\}^d$
- Graph = (vertex of polytope, edge of polytope)

<u>Conjecture</u> (0/1 polytope conjecture) [Mihail, Vazirani]
The edge expansion of the graph of a $0/1$ polytope is at least 1.

- Proven true in several important subclasses, notably matroid polytopes [Anari, Liu, Oveis Gharan, Vinzant '19]
- We gave a 0/1 polytope with small vertex expansion
- Consequence: fast uniform sampling is impossible on some 0/1 polytopes

# Summary

- Took the definition of Fastest Mixing Markov Chain
- Developed a new spectral graph theory for vertex expansion
  - Optimal Cheeger's inequality relating $\lambda_2^*$ and $\psi$




  - Cheeger-type inequalities involving $\lambda_k^*$ and vertex expansion-type quantity
- Implications about 0/1 polytope conjecture

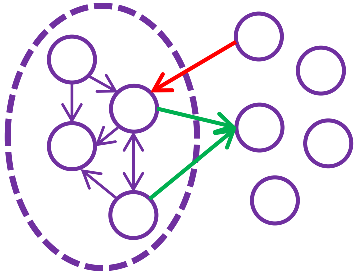
# Cheeger Inequalities for Directed Graphs and Hypergraphs using Reweighted Eigenvalues

Joint work with Lap Chi Lau and Robert Wang

# Expansion Quantities for Directed Graphs

- Directed edge conductance

$$\vec{\phi}(S) \coloneqq \frac{\min(\#\text{outgoing edges}, \#\text{incoming edges})}{\min(\mathrm{vol}(S), \mathrm{vol}(S^c))}$$



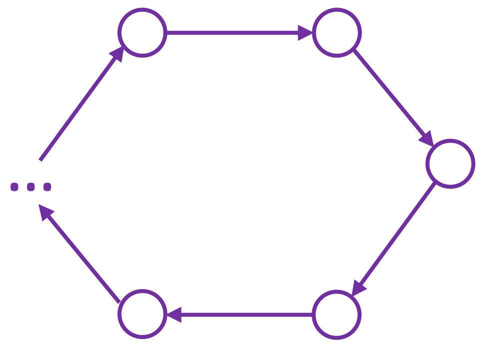
$$\vec{\phi}(S) = \frac{1}{15}$$

- Directed vertex expansion

$$\vec{\psi}(S) \coloneqq \frac{\min(\#\text{out\_neighbors}, \#\text{in\_neighbors})}{\min(|S|, |S^c|)}$$

# Spectrum of Directed Graphs…?

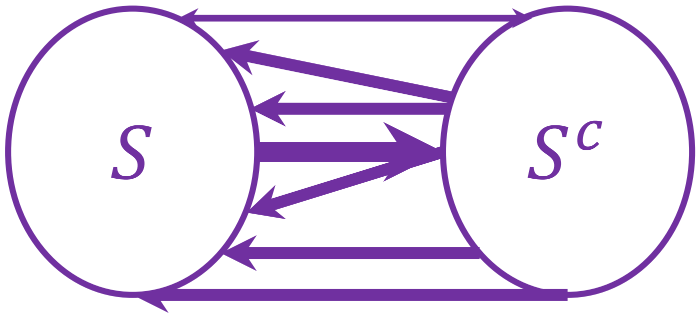
- No real eigenvalues in general

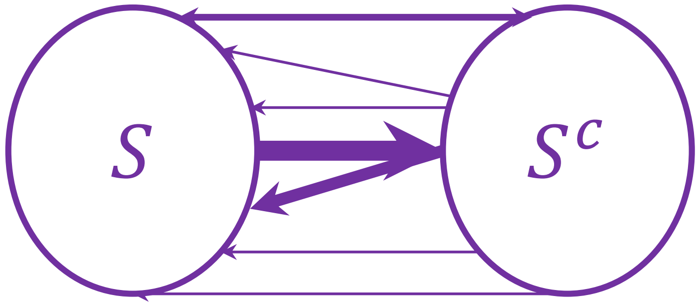



Laplacian eigenvalues are $1 - e^{2\pi i k/n}$

- Some remedies:
  - Consider associated Hermitian matrices (e.g. sum and product matrices [Fill '91], [Chung '05])
  - Nonlinear Laplacian [Yoshida '16]

# A New Remedy: Eulerian Reweighting




$$\vec{\phi}(G) \ \ (\text{or } \vec{\psi}(G)) \qquad \geq \qquad \vec{\phi}(A) = \phi\left(\frac{A + A^T}{2}\right)$$

**Find the "best" Eulerian reweighting $A$**

# Reweighted Eigenvalues for Directed Graphs

- For edge conductance:

$$\vec{\lambda}_2^{e*}(G) := \max_{A \geq 0} \lambda_2\left(D^{-\frac{1}{2}}\left(D_A - \frac{A+A^T}{2}\right)D^{-\frac{1}{2}}\right)$$

$$\begin{aligned} \text{subject to} \quad & A(u,v) = 0 & \forall uv \notin E \\ & \sum_{v \in V} A(u,v) = \sum_{v \in V} A(v,u) & \forall u \in V \\ & A(u,v) \leq w(uv) & \forall uv \in E \end{aligned}$$

Eulerian constraints on $A$

Edge capacity constraints

- For vertex expansion:

$$\vec{\lambda}_2^{v*}(G) := \max_{A \geq 0} \lambda_2\left(I - \Pi^{-\frac{1}{2}}\left(\frac{A+A^T}{2}\right)\Pi^{-\frac{1}{2}}\right)$$

$$\begin{aligned} \text{subject to} \quad & A(u,v) = 0 & \forall uv \notin E \\ & \sum_{v \in V} A(u,v) = \sum_{v \in V} A(v,u) & \forall u \in V \\ & \sum_{v \in V} A(u,v) = \pi(u) & \forall u \in V \end{aligned}$$

Eulerian constraints on $A$

Vertex capacity constraints

# Our Results

Theorem (Cheeger's Inequality for Directed Vertex Expansion) [Lau, **T.**, Wang '23]

$$\frac{\vec{\psi}(G)^2}{\log(\Delta/\vec{\psi}(G))} \lesssim \lambda_2^{v*}(G) \leq 2\vec{\psi}(G)$$

- Main application: fastest mixing general Markov chain

Theorem (Cheeger's Inequality for Directed Edge Conductance) [Lau, **T.**, Wang '23]

$$\frac{\vec{\phi}(G)^2}{\log(1/\vec{\phi}(G))} \lesssim \lambda_2^{e*}(G) \leq 2\vec{\phi}(G)$$

- Main application: certification of directed expander graphs

$\vec{\phi}(G) = \Theta(1)$ iff $\lambda_2^{e*}(G) = \Theta(1)$

# Proof Flow (same as KLT22!)

$$\gamma^{(n)} = \min_F \frac{\ell_2^2 \text{ objective of } F}{\ell_2^2 \text{ mass of } F} \quad \Rightarrow \quad \gamma^{(1)} = \min_f \frac{\ell_2^2 \text{ objective of } f}{\ell_2^2 \text{ mass of } f}$$

Gaussian Projection

Strong duality $\Uparrow$

$$\lambda_2^* = \text{reweighted eigenvalue}$$

$\Downarrow$ (1) $\ell_2^2$ to $\ell_1$ step

$$\xi = \min_h \frac{\ell_1 \text{ objective of } h}{\ell_1 \text{ mass of } h}$$

$\Downarrow$ (2) threshold rounding

$$\psi = \min_{S_t} \frac{\text{vertex boundary of } S_t}{\pi\text{−weight of } S_t}$$

$\Downarrow$

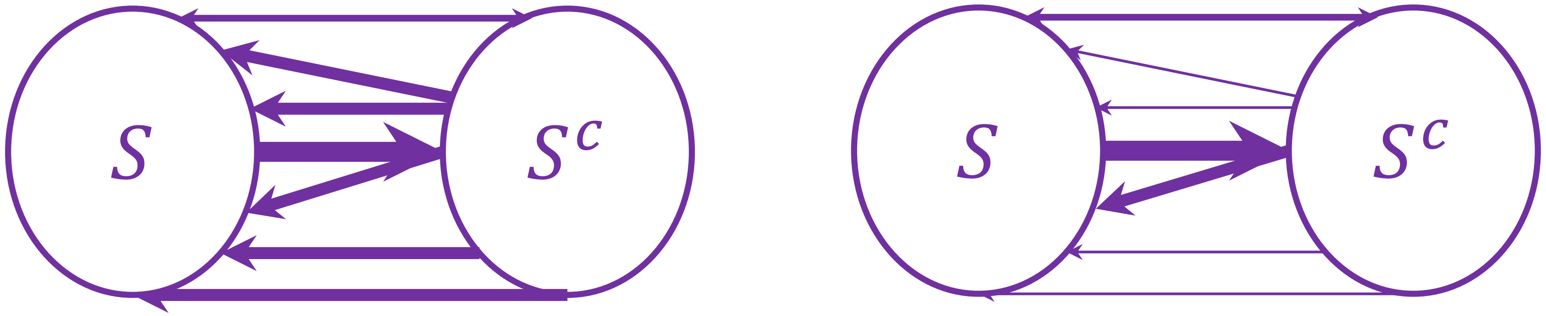
$$S_t \text{ with minimum vertex expn.}$$

Asymmetric ratio

$$\psi(S) \overset{(2)}{\leq} \xi \overset{(1)}{\leq} \sqrt{\gamma^{(1)}} \lesssim \sqrt{\gamma^{(n)} \cdot \log \alpha}$$

# Asymmetric Ratio and Large Optimal Property

- Compare the optimal reweighting objective with the trivial upper bound




$$\frac{1}{K}\sum_{uv\in E} w(u,v)\|F(u)-F(v)\|^2 \leq \sum_{uv\in E} A(u,v)\|F(u)-F(v)\|^2$$

- Projection loss is $\log K$. $K$ related to asymmetric ratio for digraphs
- Alternative proof of $\log\Delta$ projection loss for vertex expansion program [Jain, Pham, Vuong '22]

# Summary

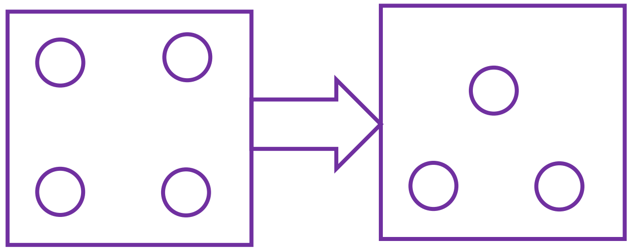
- Developed a spectral theory for directed graphs
  - Easily applied to hypergraphs
- Key idea: SDP to find “optimal” Eulerian reweighting

Main applications:

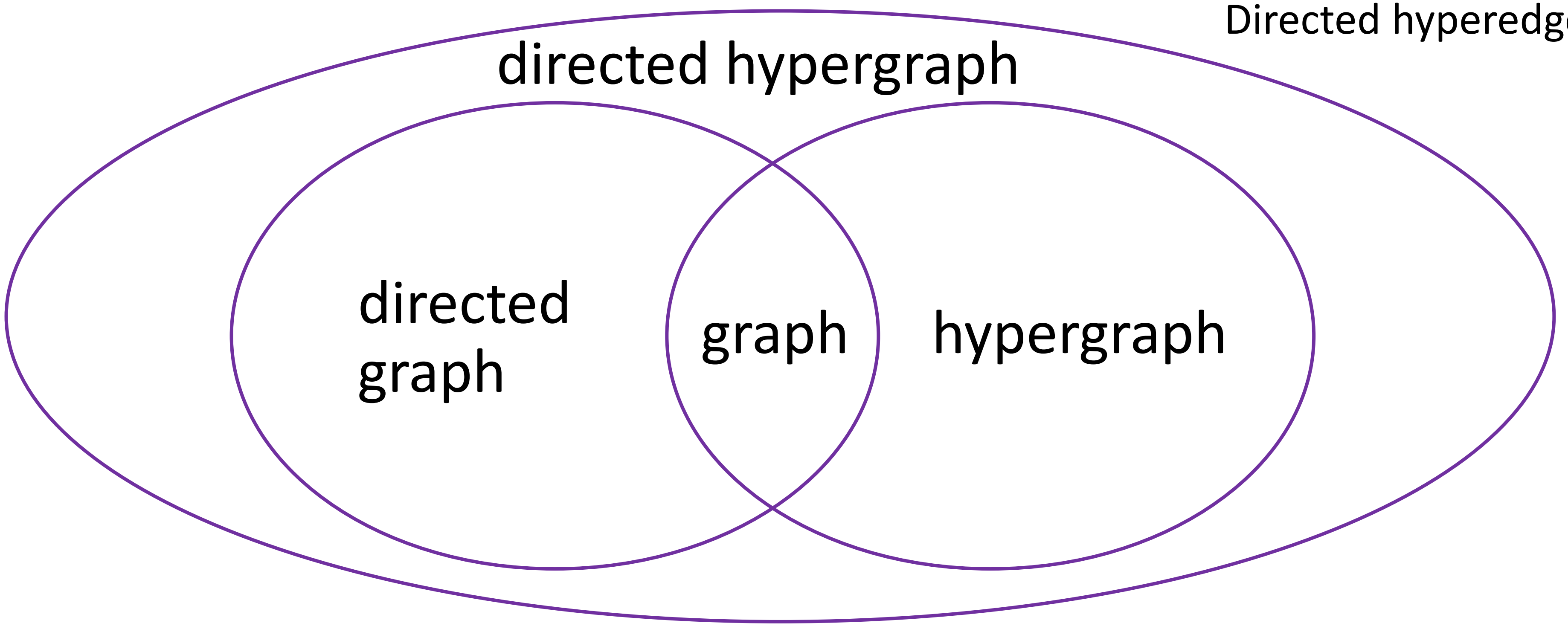
- Certification of directed expander graphs
- FMMC for non-reversible chains

Can we extend this spectral theory to a vastly more general setting?

# Directed Hypergraphs: Positive!



Directed hyperedge: $e = (e^{-}, e^{+})$




We can define reweighted eigenvalues for directed hypergraphs and obtain a Cheeger-type inequality :)

# Submodular Transformations: Negative!

- Cut function of a directed hyperedge is *submodular*
- There is a Cheeger inequality for submodular transformations by relaxing the nonlinear Laplacian [Li, Milenkovic '18], [Yoshida '19]

- Unfortunately, reweighted eigenvalues does not seem to extend to submodular transformations well :(

# Summary

- Cheeger inequality for directed hypergraphs: common generalization of all graph-like settings so far!

- Need new ideas to capture submodular transformations

# Reweighted Eigenvalue Upper Bounds for Special Families of Graphs

# Primer: Planar Separation Theorem

- Balanced separator: $S \subseteq V$ s.t. each connected component in $G[V \backslash S]$ is of size $\leq 2n/3$

- Planar graphs admit $O(\sqrt{n})$-sized separators [Lipton, Tarjan '79]

- Spectral partitioning applied recursively gives $O(\sqrt{\Delta \cdot n})$-sized separators
  - By showing that $\lambda_2' \leq O(\Delta/n)$ [Spielman, Teng '96]

# Bounded Genus and $K_h$-minor free Graphs

- Genus $g$: $G$ admits a non-crossing embedding in genus $g$ surface

- $K_h$-minor free: $G$ cannot become $K_h$ after edge/node removal and edge contraction
- There are linear-time, non-spectral algorithms that find balanced separators *given explicit embedding* [Gilbert, Hutchinson, Tarjan ’84]
- Spectral partitioning works too [Biswal, Lee, Rao ’08]
  - By upper bounding $\lambda_2'$

# Our Results

- Biswal, Lee, Rao: $\lambda_2' \leq O\left(\frac{\Delta\, g \log^2 g}{n}\right)$, $\lambda_2' \leq O\left(\frac{\Delta\, h^6 \log h}{n}\right)$
  - Implies separator size of $O(\sqrt{n\, \lambda_2'})$
- **T.**: $\lambda_2^* \leq O\left(\frac{g \log^2 g}{n}\right)$, $\lambda_2^* \leq O\left(\frac{h^6 \log h}{n}\right)$
  - Implies separator size of $O(\sqrt{n\, \lambda_2^* \cdot \log \Delta})$

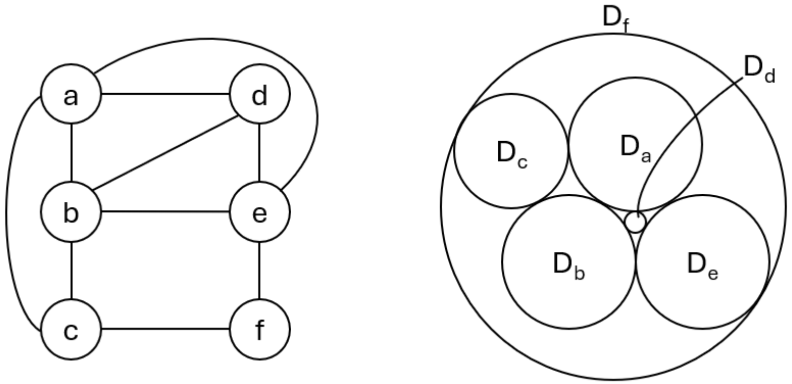
By vertex Cheeger's inequality

**Corollary: recursive spectral partitioning using reweighted eigenvalues produces smaller-sized separators**

# Planar Graphs

- For planar graphs, we can do better than the separator size of $O\left(\sqrt{n \log \Delta}\right)$ from the previous analysis!
- Idea: Use the Koebe-Andreev-Thurston kissing circle embedding to construct a solution to the 3-dim dual program $\gamma^{(3)}$ with small objective

$$\psi(S) \leq \xi \leq \sqrt{\gamma^{(1)}} \lesssim \sqrt{\gamma^{(3)}} \lesssim \sqrt{\frac{1}{n}}$$




$O(\sqrt{n})$-sized separator!

# Summary

- First spectral algorithm for $O(\sqrt{n})$-sized planar separator
- Upper bounds on $\lambda_2^*$ for vertex expansion imply balanced separator algorithms with better guarantees
- Upper bounds on $\lambda_k^*$ (applications?)

# $O\left(\sqrt{\log n}\right)$ Approximation of Expansion Quantities by adding $\ell_2^2$ Triangle Inequalities to $\lambda_2^*$, and Beyond

Joint work with Lap Chi Lau and Robert Wang

# Background: Arora-Rao-Vazirani

- $\lambda_2 = \min\limits_{f \perp deg} \frac{\sum_{uv \in E} \left(f(u) - f(v)\right)^2}{\sum_{v \in V} \deg(v) f(v)^2}$
- $\lambda_2^{\Delta} = \min\limits_{F \perp deg} \frac{\sum_{uv \in E} \|F(u) - F(v)\|^2}{\sum_{v \in V} \deg(v) \|F(v)\|^2}$

$\ell_2^2$ triangle inequalities

s.t. $\|F(u) - F(u')\|^2 + \|F(u') - F(v)\|^2 \geq \|F(u) - F(v)\|^2$

Theorem (ARV for edge conductance) [Arora, Rao, Vazirani '09]

For graphs $G$,

$$\frac{\phi(G)}{\sqrt{\log \mathrm{n}}} \lesssim \lambda_2^{\Delta}(G) \lesssim \phi(G)$$

# Our Results

- By adding $\ell_2^2$ triangle inequalities to $\lambda_2^*$, we obtain $\lambda_2^\Delta$

Theorem (ARV for generalized expansions) [Lau, **T.**, Wang '24]
For directed hypergraphs $H$,

$$\frac{\phi(H)}{\sqrt{\log \mathrm{n}}} \lesssim \lambda_2^\Delta(H) \lesssim \phi(H)$$

- Proof is adapted from ARV
- Unifies [Feige, Hajiaghayi, Lee '08], [Agarwal, Charikar, Macharychev, Macharychev '05], [Chan, Sun '18] using one single formulation!

# Background: Orthogonal Separators

- Given some set $X \subseteq \mathbb{R}^d$ of "normalized" vectors that satisfy the $\ell_2^2$ triangle inequalities
- Orthogonal separator is a random subset $S \subseteq X$ such that:
  - $\Pr[x \in S] \propto \|x\|^2$
  - If $x$ and $y$ are "far away", then probability that both are included is "exceptionally" small
  - If $x$ and $y$ are "close", then probability that one is in $S$ while the other is not (i.e. $xy$ cut by $S$) is small
- Used to approximate unique games [Chlamtac, Makarychev, Makarychev '06] and various expansion problems [Bansal et al. '14], [Louis, Makarychev '14a]
- "Hypergraph" orthogonal separator [Louis, Makarychev '14b]

# Our Results

- Let $\phi_k(H)$ be k-way expansion of hypergraph $H$
- By adding $\ell_2^2$ triangle inequalities to $\sigma_k^*$, we obtain $\lambda_k^\Delta$

<u>Theorem</u> (LM for $k$-way expansion) [T., unpublished]
For hypergraphs $H$, $\lambda_k^\Delta(H) \lesssim \phi_k(H)$ and

$$\phi_{(1-\epsilon)k}(H) \lesssim_\epsilon k \log k \log\log k \cdot \sqrt{\log n} \cdot \lambda_k^\Delta(H).$$

- Proof is adapted from [Louis, Makarychev '14b]
- First true approximation of $k$-way expansions without square-root loss

# Summary

- By adding $\ell_2^2$ triangle inequalities to $\lambda_2^*$ and $\sigma_k^*$, we can adapt existing techniques to obtain approximation algorithms for expansion quantities

# Concluding Remarks

- We developed the reweighted eigenvalue framework
- Obtained a spectral theory for a general class of expansion problems, with many nice applications
- Higher reweighted eigenvalues also useful

- Some concrete open problems about the theory
- Future research directions: local algorithms, applications of optimal primal reweighting, fast algorithms, …

# THANK YOU!

# Questioning Period