Cheeger Inequalities for Directed Graphs and Hypergraphs Using Reweighted Eigenvalues

Kam Chuen (Alex) Tung
PhD Candidate, University of Waterloo
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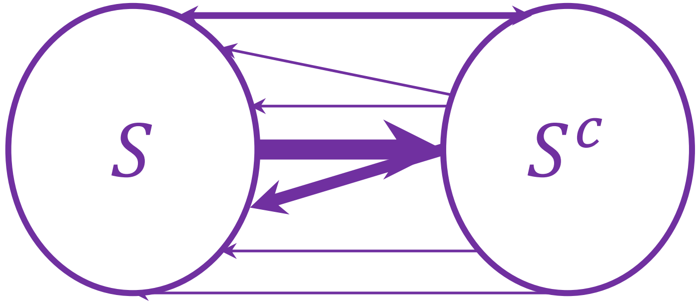

Joint work with Lap Chi Lau (U Waterloo) and Robert Wang (U Waterloo)

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Cheeger’s inequality

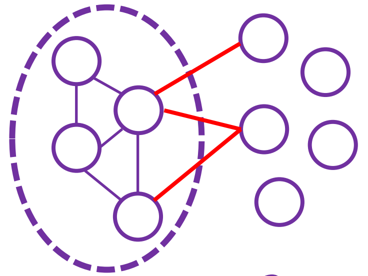
- $G = (V, E)$ undirected
- Conductance of graph: $\phi(G) := \min_{vol(S) \le vol(V)/2} \frac{|\delta(S)|}{vol(S)}$

#edges across S

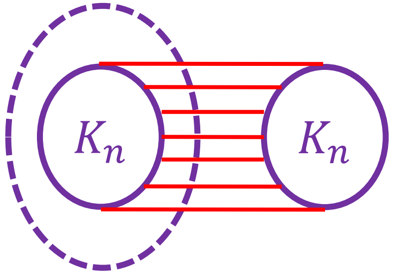
Total degree of S

$\mathcal{A} := D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$

- Eigenvalues of Laplacian $\mathcal{L} := I - \mathcal{A}$ are $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 2$



$\phi(S) = \frac{3}{13}$



$\phi(G) = \Theta(\frac{1}{n})$

<u>Theorem</u> [Cheeger ’70, Alon, Milman ’85, Alon ’86]

$$\frac{\lambda_2}{2} \le \phi \le \sqrt{2\,\lambda_2}$$

- Generalizations abound [Trevisan ’09], [LOT ’12], [LRTV ’12], [KLLOT ’13]

λ_2 and mixing time

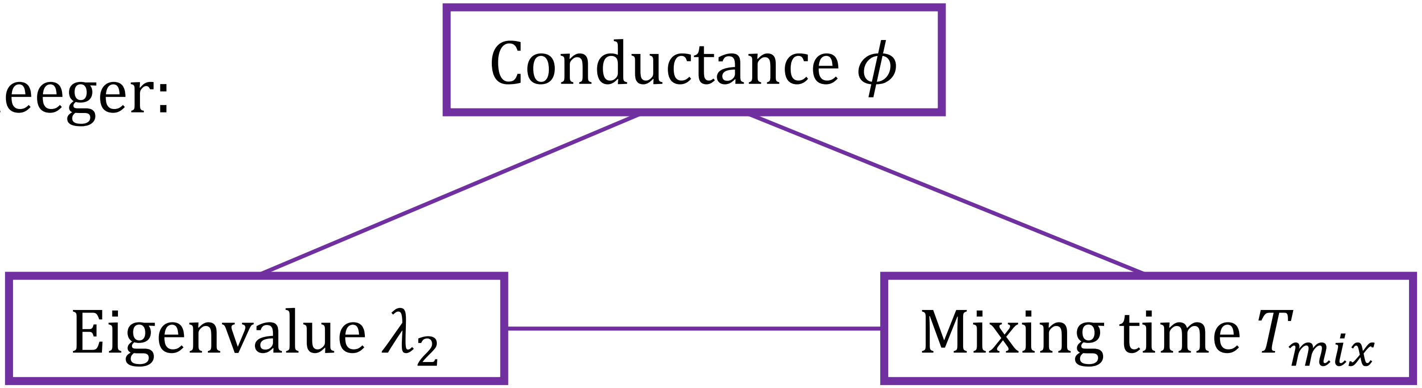
Go to a random neighbor of the current vertex u

- Let P be the canonical random walk on G
- Mixing time is roughly proportional to $1/\lambda_2$:

time needed to get $(1/e)$−close to s.d. π

$$\frac{1}{\lambda_2} \lesssim T_{mix}(P) \lesssim \frac{1}{\lambda_2} \cdot \log \frac{1}{\pi_{min}}$$

- Summary of classical Cheeger:



Question: Spectral theory for directed graphs? Hypergraphs?

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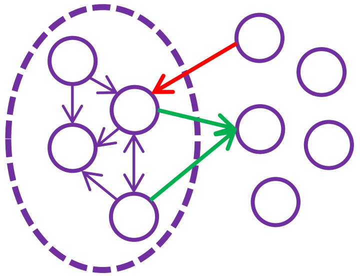
Directed quantities



$\vec{\phi}(S) = \frac{1}{8}$

- Directed *edge conductance*:

Edges from S to S^c

Edges from S^c to S

$$\vec{\phi}(G) \coloneqq \min_{vol(S) \le vol(V)/2} \frac{\min(w(E(S, S^c)), w(E(S^c, S)))}{vol(S)}$$

Sum of outdegrees in S

- Directed *vertex expansion*: $\vec{\psi}(G) \coloneqq \min_{\pi(S) \le \pi(V)/2} \frac{\min(\pi(N(S)), \pi(N(S^c)))}{\pi(S)}$

(A sample of) past work

[Fill '94], [Chung '05] Cheeger constant $h(G)$

$h(G)$: π_{sd}-weighted conductance

- The reweighted graph F with $f(u,v) = \pi_{sd}(u)P(u,v)$ is Eulerian.

  \+ Cheeger Inequality: $\frac{\lambda_2(\tilde{L})}{2} \leq h(G) \leq \sqrt{2\lambda_2(\tilde{L})}$

  \+ $\lambda_2(\tilde{L})$ has relation to mixing time

  − π_{sd} can be erratic. Not so useful for graph algorithms

[Yoshida '19] Cheeger inequality for non-linear Laplacian

\+ Cheeger Inequality: $\frac{\lambda_G}{2} \leq \vec{\phi}(G) \leq 2\sqrt{\lambda_G}$

− Not poly-time solvable

Edge energies

$$\lambda_G = \inf_{x: \sum \deg(u)x(u)=0} \frac{\sum_{uv\in E} w(u,v)\left(\left(x(u)-x(v)\right)^+\right)^2}{\sum_{u\in V} \deg(u)x(u)^2}$$

Normalization

Bottom line

The *nonlinearity/asymmetry* (of the directed quantities and the associated Laplacians) makes spectral theory difficult!

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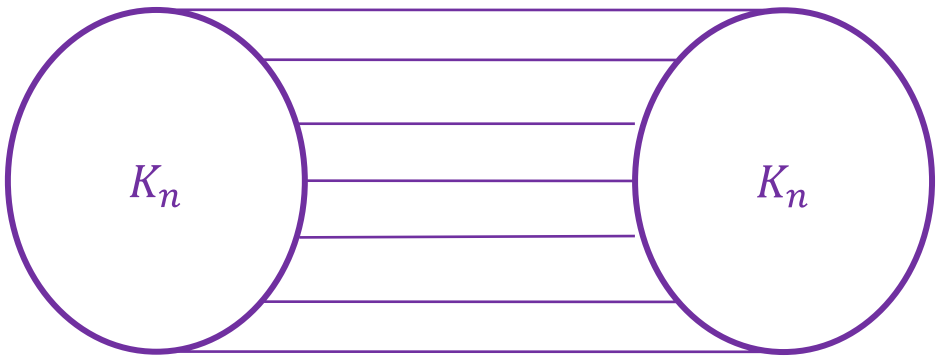
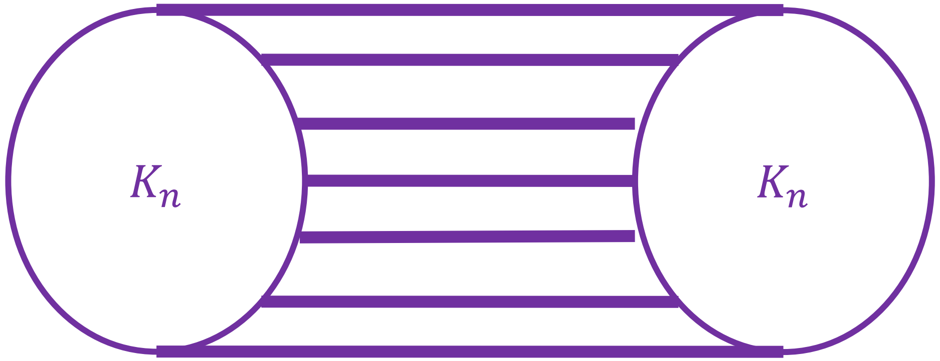
Reweighting example





Mixing time is $\Theta(n)$

Mixing time is $\Theta(1)$

Reweighted eigenvalue

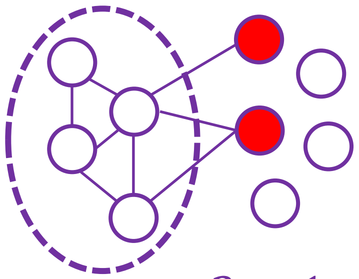
- [BDX ’04] “Fastest mixing Markov chain”
- Key idea: eigenvalue λ_2 as proxy for mixing time

- It is the following program:

$\lambda_2(I - P)$

$$
\begin{aligned}
\lambda_2^*(G) \;:=\; \max_{P \geq 0} \quad & 1 - \alpha_2(P) & \\
\text{subject to} \quad & P(u,v) = P(v,u) = 0 & \forall uv \notin E \\
& \sum_{v \in V} P(u,v) = 1 & \forall u \in V \\
& \pi(u)P(u,v) = \pi(v)P(v,u) & \forall uv \in E.
\end{aligned}
$$

Vertex expansion

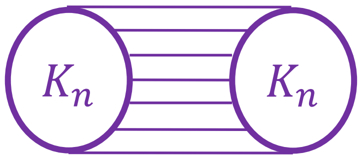


$\psi(S) = \frac{2}{4} = \frac{1}{2}$

- $G = (V, E)$ undirected
- Vertex expansion of graph: $\psi(G) \coloneqq \min_{|S| \le |V|/2} \frac{|N(S)|}{|S|}$
- Can define π-weighted version

#neighbors of S

Size of S



$\psi(G) = \Theta(1)$

<u>Theorem</u> [Olesker-Taylor, Zanetti '21, KLT '22]

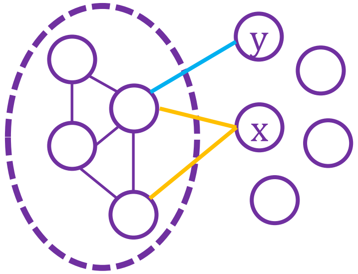
$$\frac{\lambda_2^*}{2} \le \psi \lesssim \sqrt{\lambda_2^* \cdot \log \Delta}$$

- This spectral theory encompasses: fastest mixing time, "generalizations"

Question: Reweighted eigenvalues for directed graphs? Hypergraphs?

Reweighting as "best" certificate

- In a feasible reweighting $Q = \Pi P, \sum_j Q(i,j) = \pi(i)$
- Amount of edge weight from S to $S^c \leq \pi(N(S))$



regular Cheeger

- $\lambda_2\left(\Pi^{-\frac{1}{2}}(\Pi - Q)\Pi^{-\frac{1}{2}}\right) \lesssim \phi(Q) \leq \psi(G)$

reweighting

λ_2^* is a (computable) proxy for ϕ^* to lower bound $\psi(G)$

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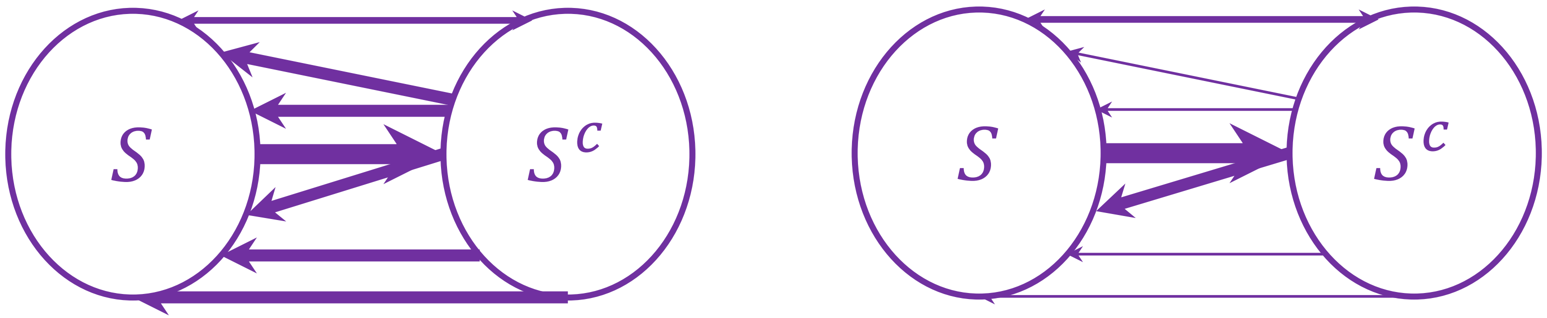
Certificates for expansion

- Want a class of reweighted subgraphs as certificates
- Want the certificates to be *computable*

Key idea: consider Eulerian reweightings



$$\vec{\phi}(G) \text{ or } \vec{\psi}(G) \quad \geq \quad \vec{\phi}(A) = \phi\left(\frac{A + A^T}{2}\right)$$

Reweighted eigenvalue for directed graphs

- Edge:

$$\vec{\lambda}_2^{e*}(G) := \max_{A \geq 0} \lambda_2\left(D^{-\frac{1}{2}}\left(D_A - \frac{A + A^T}{2}\right)D^{-\frac{1}{2}}\right)$$

$$\begin{aligned} \text{subject to } & A(u,v) = 0 && \forall uv \notin E \\ & \sum_{v \in V} A(u,v) = \sum_{v \in V} A(v,u) && \forall u \in V \\ & A(u,v) \leq w(uv) && \forall uv \in E \end{aligned}$$

Eulerian constraints on A

Edge capacity constraints

- Vertex:

$$\vec{\lambda}_2^{v*}(G) := \max_{A \geq 0} \lambda_2\left(I - \Pi^{-\frac{1}{2}}\left(\frac{A + A^T}{2}\right)\Pi^{-\frac{1}{2}}\right)$$

$$\begin{aligned} \text{subject to } & A(u,v) = 0 && \forall uv \notin E \\ & \sum_{v \in V} A(u,v) = \sum_{v \in V} A(v,u) && \forall u \in V \\ & \sum_{v \in V} A(u,v) = \pi(u) && \forall u \in V \end{aligned}$$

Eulerian constraints on A

Vertex capacity constraints

Main results

Theorem 1 [LTW '22] For arbitrary w,

$$\frac{\lambda_2^{e*}}{2} \leq \vec{\phi} \lesssim \sqrt{\lambda_2^{e*} \cdot \log \frac{1}{\lambda_2^{e*}}}$$

Theorem 2 [LTW '22] For arbitrary π,

$$\frac{\lambda_2^{v*}}{2} \leq \vec{\psi} \lesssim \sqrt{\lambda_2^{v*} \cdot \log \frac{\Delta}{\lambda_2^{v*}}}$$

Some consequences

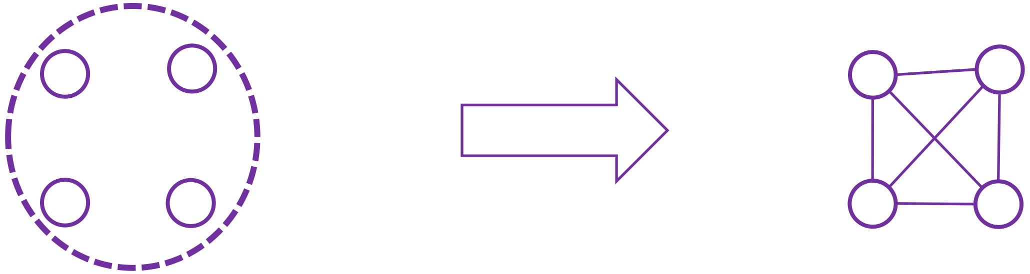
- Cheeger cuts
  - O(SDP) time compute sparse cuts with these Cheeger-type guarantees
- Certifying constant $\vec{\phi}$
  - $\vec{\phi} = \Theta(1)$ iff $\lambda_2^{e*} = \Theta(1)$
- Fastest mixing Markov chain on directed graphs
  - $\vec{\psi}$ is the only obstacle to fastest mixing

Corollary 3 [LTW '22] For arbitrary distribution π,

$$\frac{1}{\vec{\vec{\psi}}} \cdot \frac{1}{\log\left(\frac{1}{\pi_{min}}\right)} \lesssim T_{mix}^* \lesssim \frac{1}{\vec{\vec{\psi}}^2} \cdot \log\frac{\Delta}{\vec{\vec{\psi}}} \cdot \log\frac{1}{\pi_{min}}$$

Hypergraphs

- Same spirit, reweight the “clique graph”:



- Recovers and improves some results of [Louis]

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Rewriting the program

- We will look at λ_2^{e*}. Rayleigh quotient + von Neumann minimax ⇒

$$\vec{\lambda}_2^{e*}(G) := \min_{f:V\to\mathbb{R}^n} \max_{A\geq 0} \frac{1}{2} \sum_{uv\in E} A(u,v)\cdot\|f(u)-f(v)\|^2$$

$$\text{subject to } A(u,v) = 0 \qquad \forall uv \notin E$$

$$\sum_{v\in V} A(u,v) = \sum_{v\in V} A(v,u) \qquad \forall u \in V$$

$$A(u,u) \leq w(uv) \qquad \forall uv \in E$$

$$\sum_{v\in V} d_w(v)\cdot f(v) = \vec{0}$$

$$\sum_{v\in V} d_w(v)\cdot\|f(v)\|^2 = 1.$$

Eulerian constraints on A

Edge capacity constraints

Normalization constraints on f

1D program

- Define $\lambda_e^{(k)}$ to be

$$\min_{f:V\to\mathbb{R}^k} \max_{A} \frac{1}{2}\sum_{ij\in E} A(i,j)\|f(i)-f(j)\|^2$$

where A is the set of Eulerian reweightings subject to edge capacity constraints

- We would like to project the original n-dimensional solution for $\lambda_2^{e*} = \lambda_e^{(n)}$ to a 1D solution for $\lambda_e^{(1)}$

With capacity constraints

Proposition [JPV'22, LTW '22] (Large optimal property)
If the original graph can be "covered" by M reweightings, then random projection loses a factor of $\log O(M)$

Asymmetric ratio and projection

- Intuition: if G is Eulerian then $M = 1$; if G has very unbalanced cut then M should be large.
- Define asymmetric ratio as $\alpha \coloneqq \max_S \frac{w(E(S,S^c))}{w(E(S^c,S))}$
  - Using Hoffman's circulation lemma, can show $M = O(\alpha)$ for $\vec{\phi}$
  - Moreover, $\alpha \le \frac{1}{\vec{\phi}} \lesssim \frac{1}{\lambda_2^{e*}}$
  - Therefore, projection loss is $O\left(\log \frac{1}{\lambda_2^{e*}}\right)$
- For directed vertex expansion: $M = O(\Delta \cdot \alpha)$
- For undirected vertex expansion: M = $O(\Delta)$
- For hypergraph conductance: M = $O(r)$

ℓ_1 program

Definition 3.19 (ℓ_1 Version of $\vec{\lambda}_e^{(1)}$). *Given an edge-weighted directed graph $G = (V, E, w)$, let*

$$
\begin{aligned}
\eta_e(G) := \min_{f:V\to\mathbb{R}} \max_{A\geq 0} \quad & \frac{1}{2}\sum_{uv\in E} A(u,v)\cdot|f(u)-f(v)| \\
\text{subject to} \quad & A(u,v) = 0 && \forall uv \notin E \\
& \sum_{v\in V} A(u,v) = \sum_{v\in V} A(v,u) && \forall u \in V \\
& A(u,u) \leq w(uv) && \forall uv \in E \\
& \sum_{v\in V} d_w(v)\cdot f(v) = 0 \\
& \sum_{v\in V} d_w(v)\cdot |f(v)| = 1.
\end{aligned}
$$

Now ℓ_1 metric

Now ℓ_1 normalization

- We prove that $\eta_e \lesssim \sqrt{\lambda_e^{(1)}}$ by constructing ℓ_1 solutions from ℓ_2 solutions

Threshold rounding

- LP dual of the 1D ℓ_1 program:

$$\begin{aligned}\xi_e(G) := \min_{f:V\to\mathbb{R}} \min_{\substack{q:E\to\mathbb{R}_{\geq 0} \\ r:V\to\mathbb{R}}} \quad & \sum_{uv\in E} w(uv)\cdot q(uv) \\ \text{subject to} \quad & q(uv) \geq |f(u)-f(v)| - r(u) + r(v) \qquad \forall uv\in E \\ & \sum_{v\in V} d_w(v)\cdot f(v) = 0 \\ & \sum_{v\in V} d_w(v)\cdot |f(v)| = 1.\end{aligned}$$

- Given a feasible solution (f, r) to the ℓ_1 program, with objective OBJ
- Use threshold rounding to obtain cut with $\vec{\phi}(S) \leq O(OBJ)$
  - Consider $(f+r)^+, (f+r)^-, (f-r)^+, (f-r)^-$

Recap

Reweighted eigenvalue λ_2^{e*}

$\Downarrow$ Rayleigh + minimax

n-dimensional program $\lambda_e^{(n)}$

$\Downarrow$ Large optimal property, $M = O(\alpha)$

1-dimensional program $\lambda_e^{(1)}$

$\Downarrow$ Construct ℓ_1 solution, sqrt loss

1D ℓ_1 program

$\Downarrow$ Threshold rounding

Cut S, $\vec{\phi}(S) \lesssim \sqrt{\lambda_2^* \cdot \log \alpha}$

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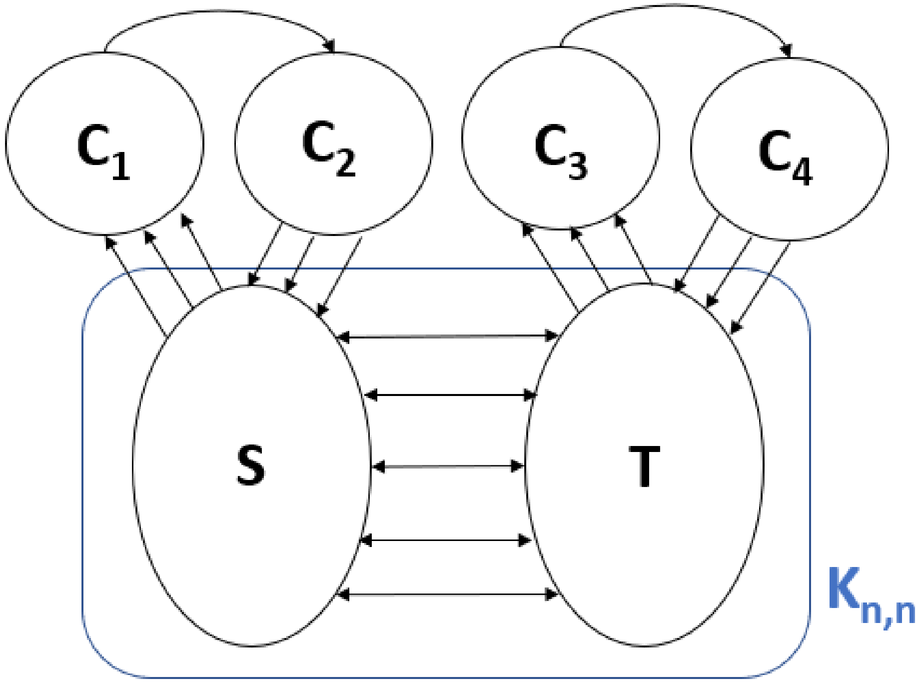
"Generalizations"

- "Bipartite Cheeger" [Trevisan '09] does not extend naturally

λ_n and bipartiteness

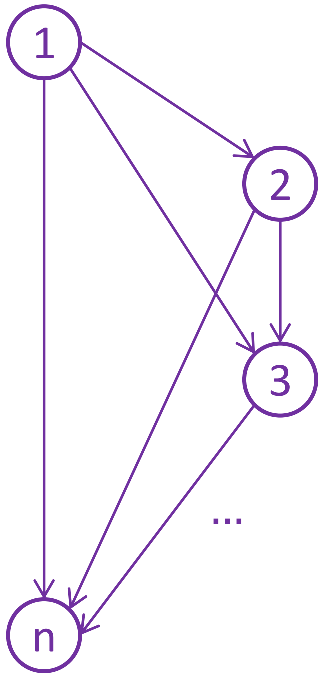


$2 - \lambda_n^{e*} = O(\frac{1}{n^2})$ but no near-bipartite sparse cut

“Generalizations”

- Higher-order Cheeger [LOT ‘12, LRTV ‘12] does not either…

λ_k and k-way conductance



$\lambda_k^{e*} = 0$ but $\vec{\phi}_k$ is large for $k \geq 3$

"Generalizations"

λ_2, λ_k and conductance

- But improved Cheeger [KLLOT '13] has a directed analogue

<u>Theorem 3</u> [LTW '22] (Informal)
If λ_k^{e*} is large for some small k, then (up to log factors) we can upper bound $\vec{\phi}$ by $O(\lambda_2^{e*})$ instead of $O(\sqrt{\lambda_2^{e*}})$.

An alternative viewpoint

Reweighted eigenvalue λ_2^{e*}

⇓ Rayleigh + minimax

n-dimensional program $\lambda_e^{(n)}$

⇓ Large optimal property, $M = O(\alpha)$

1-dimensional program $\lambda_e^{(1)}$

⇓ Construct ℓ_1 solution, sqrt loss

1D ℓ_1 program

⇓ Threshold rounding

Cut S, $\vec{\phi}(S) \lesssim \sqrt{\lambda_2^* \cdot \log \alpha}$

$\min_S \vec{\phi}(S)$

⇑ $\{0, 1\}^n$ to $[0, 1]^n$; integral

1D ℓ_1 *unreweighted* program

⇑ **Lossless Symmetrization**

1D ℓ_1 reweighted program

⇑ Lifting to ℓ_2; sqrt loss

1D ℓ_2 reweighted program

⇖ Convexify (to n-dimensional program $\lambda_e^{(n)}$)

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Takeaways

- A spectral theory based on *Eulerian reweighting*
- *Unified* theory for all graph/hypergraph settings -> all reduce to classical theory for edge conductance

- Exciting to see further developments!

Some further questions

Applications:

- Fast (e.g. almost-linear time) algorithms? ← [LTW ‘23]: Yes!
- Concrete practical applications?

Theory:

- How to formulate bipartite/higher-order Cheeger for directed graphs?
- Are the log terms in the Cheeger inequalities tight?
- Connections with submodular transformations (à la Yoshida)? ←

The end