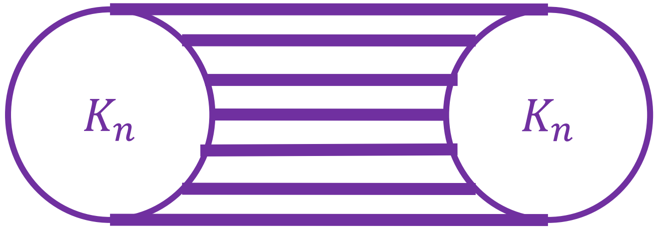
Cheeger Inequalities for Vertex Expansion and Reweighted Eigenvalues



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Joint work with:

Tsz Chiu Kwok (Shanghai U of Finance and Economics)

Lap Chi Lau (U Waterloo)

Outline

- Classical Cheeger's inequality
- Vertex expansion
- Reweighted eigenvalues
- Our results
- Proof ideas
- Summary

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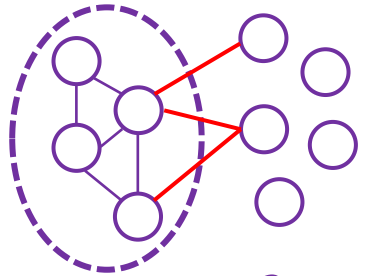
Cheeger's inequality



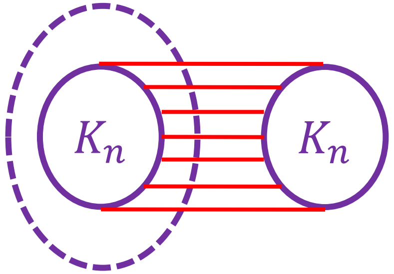
- $G = (V, E)$ undirected
- Conductance of graph: $\phi(G) := \min_{vol(S) \le vol(V)/2} \frac{|\delta(S)|}{vol(S)}$

#edges across S

Total degree of S

$\mathcal{A} := D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$

- Eigenvalues of Laplacian $\mathcal{L} := I - \mathcal{A}$ are $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 2$



<u>Theorem</u> [Cheeger '70, Alon, Milman '85, Alon '86]

$$\frac{\phi^2}{2} \le \lambda_2 \le 2\,\phi$$

- Generalizations abound [Trevisan '09], [LOT '12], [LRTV '12], [KLLOT '13]

λ_2 and mixing time

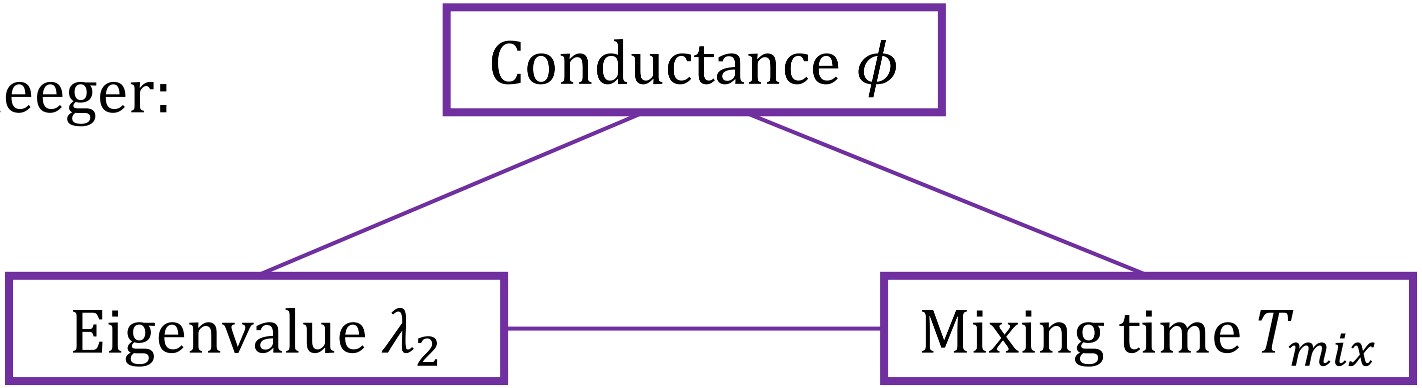
Go to a random neighbor of the current vertex u

- Let P be the canonical random walk on G
- Mixing time is roughly proportional to $1/\lambda_2$:

time needed to get $(1/e)$−close to s.d. π

$$\frac{1}{\lambda_2} \lesssim T_{mix}(P) \lesssim \frac{1}{\lambda_2} \cdot \log \frac{1}{\pi_{min}}$$

- Summary of classical Cheeger:



Question: is there an analogous theory for vertex expansion?

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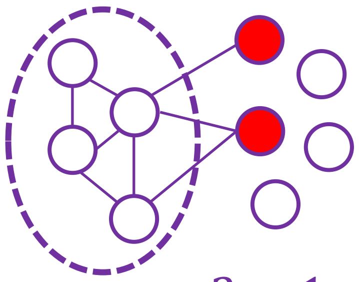
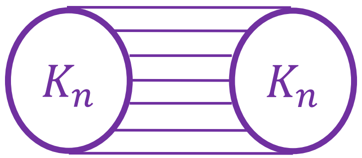
Vertex expansion



$\psi(S) = \frac{2}{4} = \frac{1}{2}$



$\psi(G) = \Theta(1)$

- $G = (V, E)$ undirected
- Vertex expansion of graph: $\psi(G) := \min_{|S| \leq |V|/2} \frac{|N(S)|}{|S|}$

#neighbors of S

Size of S

- Past work on ψ include:
  - Unnormalized Laplacian λ_2' [Tanner '84], [Alon, Milman '85]
  - Spectral quantity λ_∞ [Bobkov, Houdré, Tetali '00]
  - SDP relaxation sdp_∞ [Louis, Raghavendra, Vempala '13]
  - Extension of ARV [Feige, Hajiaghayi, Lee '08]
  - Spectral hypergraph theory [Louis '15], [CLTZ '18]

Lots of past work, but no nice spectral theory for ψ :(

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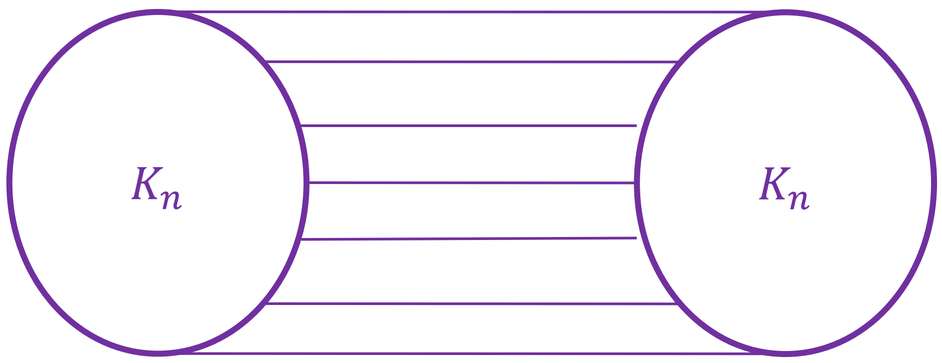
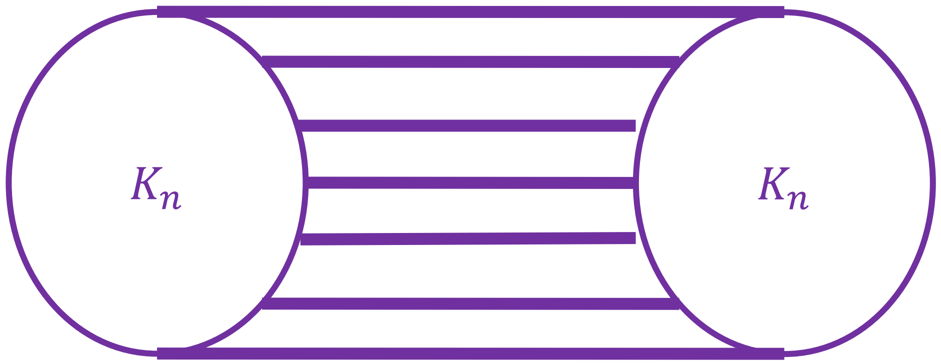
Reweighting example



Mixing time is $\Theta(n)$



Mixing time is $\Theta(1)$

Reweighted eigenvalue

- [BDX '04] "Fastest mixing Markov chain"
- Key idea: eigenvalue λ_2 as proxy for mixing time

- It is the following program:

$\lambda_2(I - P)$

$$\begin{aligned}
\lambda_2^*(G) \;:=\; \max_{P \geq 0} \quad & 1 - \alpha_2(P) \\
\text{subject to} \quad & P(u,v) = P(v,u) = 0 && \forall uv \notin E \\
& \sum_{v \in V} P(u,v) = 1 && \forall u \in V \\
& \pi(u)P(u,v) = \pi(v)P(v,u) && \forall uv \in E.
\end{aligned}$$

Cheeger's inequality for ψ

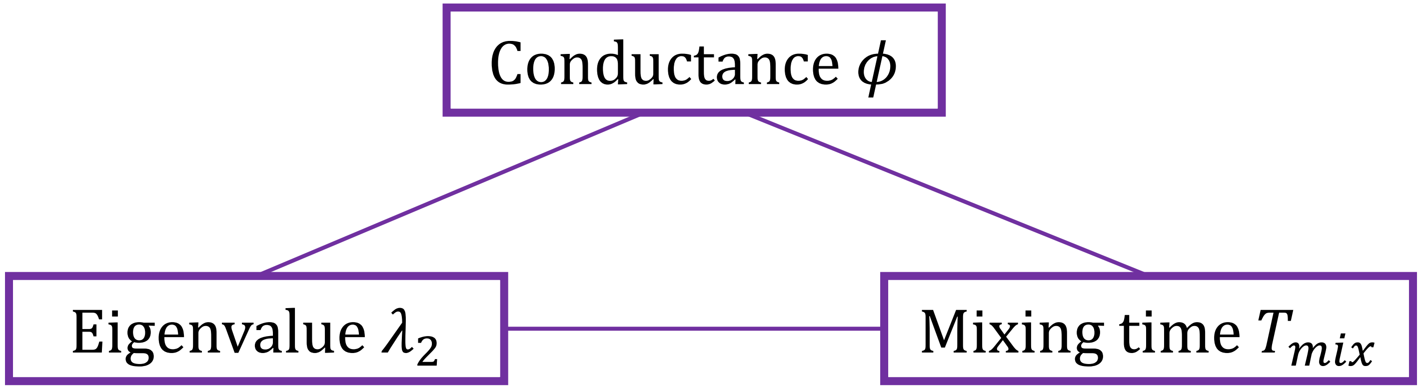
- Relation between λ_2^* and ψ

<u>Theorem</u> [Olesker-Taylor, Zanetti '22] For π uniform,

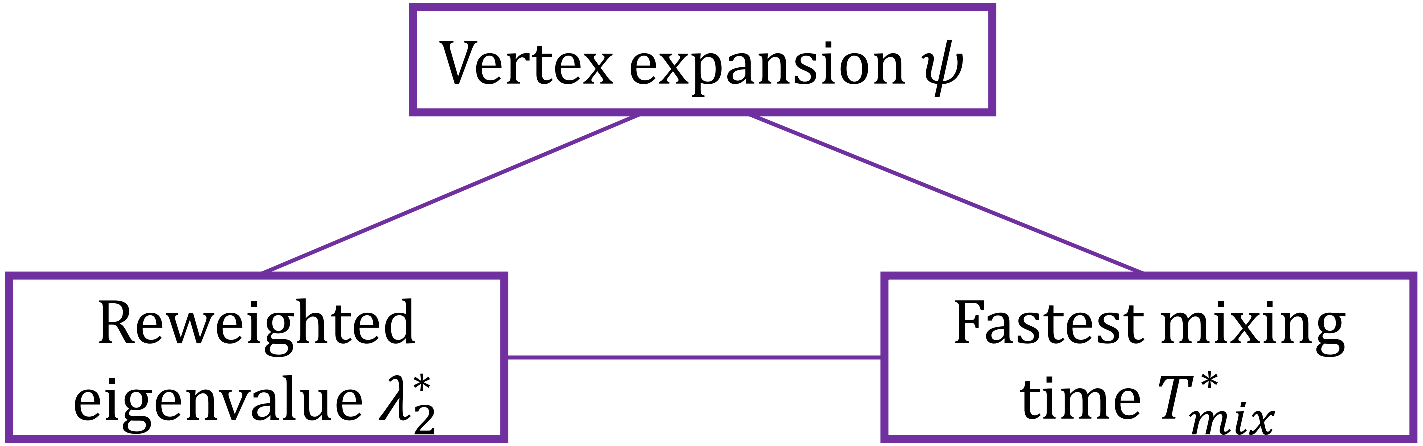
$$\frac{\psi(G)^2}{\log|V|} \lesssim \lambda_2^*(G) \lesssim \psi(G).$$

A new vertex spectral theory

- Edge:



- Vertex:



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1) Cheeger's inequality for ψ, v.2

<u>Theorem</u> [Olesker-Taylor, Zanetti '22] For π uniform,

$$\frac{\psi(G)^2}{\log|V|} \lesssim \lambda_2^*(G) \lesssim \psi(G).$$

- They left open the following questions:
  - arbitrary distribution π?
  - $\log|V|$ replaced by $\log d$? ([LRV '13]: SSE-hard to go beyond)

d: max degree

1) Cheeger's inequality for ψ, v.2

- We answered the questions in the affirmative:

<u>Theorem</u> [Kwok, Lau, T. '22] For arbitrary π,

$$\frac{\psi(G)^2}{\log d} \lesssim \lambda_2^*(G) \lesssim \psi(G).$$

- Furthermore, the log-dependence on d is optimal (not just SSE-optimal)

2) Generalizations of Cheeger's inequalities

Eigenvalues (edge)	Reweighted eigenvalues (vertex)

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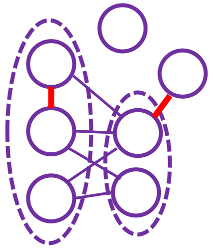

Eigenvalues (edge)	Reweighted eigenvalues (vertex)
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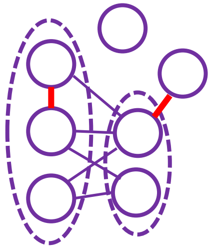
2) Generalizations of Cheeger's inequalities



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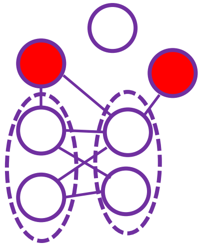

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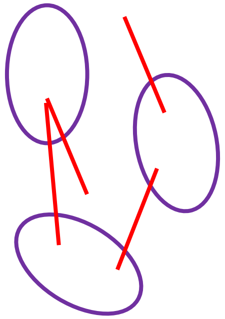
<u>Higher-order Cheeger</u> [LOT ’12, LRTV ’12] $\lambda_k(G) \lesssim \phi_k(G) \lesssim \mathrm{k}^2\sqrt{\lambda_k(G)}$ Relates λ_k to “k-way conductance”	



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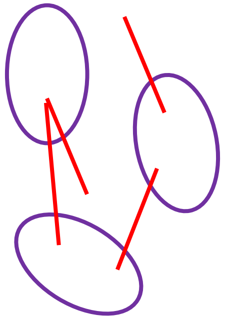

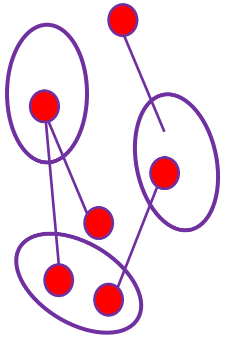

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3) 0/1 polytope with torpid mixing

- A 0/1 polytope is a polytope with vertices in $\{0,1\}^d \subseteq \mathbb{R}^d$

Conjecture (0/1 polytope conjecture)
The graph of any 0/1 polytope has edge expansion ≥ 1.

- If true implies fast sampling using random walks

- On the sampling side, we obtain the following negative evidence:

Theorem [Kwok, Lau, T. '22]
For fixed k and large enough n, there is a 0/1 polytope Q with $O\left(n^k\right)$ vertices and $\psi(Q) \lesssim O_k\left(\frac{1}{n^{k-2}}\right)$.

- As a corollary, for all $\epsilon > 0$ there is a 0/1-polytope with *fastest* mixing time $\Omega(|V|^{1-\epsilon})$ to the uniform distribution

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Dual of λ_2^* program [Roch '05]

- Primal:

$$\begin{aligned} \lambda_2^*(G) := \max_{P \geq 0} \quad & 1 - \alpha_2(P) \\ \text{subject to} \quad & P(u,v) = P(v,u) = 0 && \forall uv \notin E \\ & \sum_{v \in V} P(u,v) = 1 && \forall u \in V \\ & \pi(u)P(u,v) = \pi(v)P(v,u) && \forall uv \in E. \end{aligned}$$

Fractional matching

Rayleigh quotient + minimax + LP duality

- Dual:

$$\begin{aligned} \gamma(G) := \min_{f:V\to\mathbb{R}^n,\ g:V\to\mathbb{R}_{\geq 0}} \quad & \sum_{v\in V} \pi(v) g(v) \\ \text{subject to} \quad & \sum_{v\in V} \pi(v) \|f(v)\|^2 = 1 \\ & \sum_{v\in V} \pi(v) f(v) = \vec{0} \\ & g(u) + g(v) \geq \|f(u) - f(v)\|^2 && \forall uv \in E. \end{aligned}$$

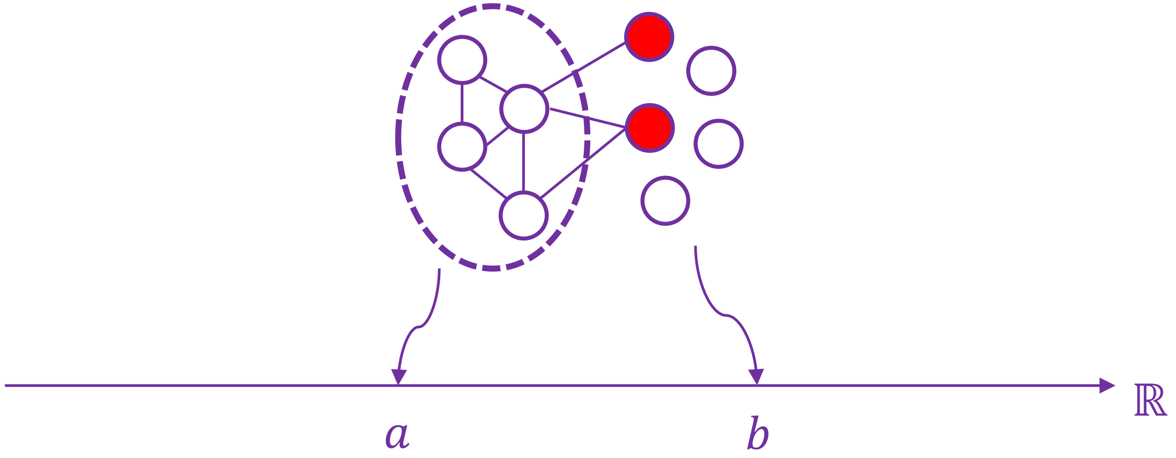
Fractional vertex cover

$\gamma^{(k)}(G)$ if $f: V \to \mathbb{R}^k$

Normalization constraints

Easy direction

- Every vertex cut S can be realized as a two-point embedding



- Cover all crossing edges using $g(u) = [u \in N(S)] \cdot (a - b)^2$
- Check that $\sum_u \pi(u) g(u) \leq 2\, \psi(S)$

Hard direction: overview

<u>Theorem</u> [Olesker-Taylor, Zanetti '22] For π uniform,

$$\frac{\psi(G)^2}{\log|V|} \lesssim \lambda_2^*(G) \lesssim \psi(G).$$

1. Take dual: $\gamma^{(n)}(G) = \lambda_2^*(G)$
2. J-L lemma: $\gamma^{(1)}(G) \lesssim \log|V| \cdot \gamma^{(n)}(G)$
3. Round 1D solution to matching conductance

<u>Theorem</u> [Kwok, Lau, T. '22] For arbitrary π,

$$\frac{\psi(G)^2}{\log d} \lesssim \lambda_2^*(G) \lesssim \psi(G).$$

1. Take dual: $\gamma^{(n)}(G) = \lambda_2^*(G)$
2. Gaussian projection: $\gamma^{(1)}(G) \lesssim \log d \cdot \gamma^{(n)}(G)$

   [Jain, Pham, Vuong '22] showed the same for π uniform
3. Round 1D solution to "directed vertex expansion"

Key intermediary is "directed program"

Directed program

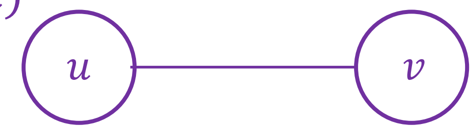
- Original "vertex cover" constraint:

$$g(u) + g(v) \geq \|f(u) - f(v)\|^2 \;\; \forall\, uv \in E$$

- Direct all edges "appropriately"
- Constraint changed to $g(v) \geq \|f(u) - f(v)\|^2 \;\; \forall u \to v$

Fact: $\gamma^{(k)}(G) \leq \vec{\gamma}^{(k)}(G) \leq 2\gamma^{(k)}(G)$

$g(v) \geq g(u)$



$g(u) + g(v) \geq \|f(u) - f(v)\|^2$

$g'(u) \coloneqq 2g(u) \;\; \forall u \in V$

$g'(v) \geq \|f(u) - f(v)\|^2$

Better (analysis of) projection

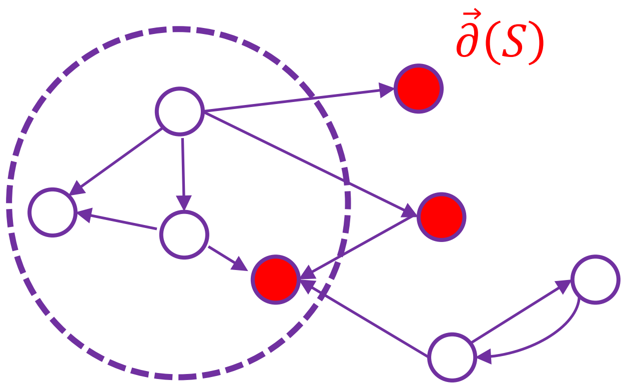
$\Pi: \mathbb{R}^n \to \mathbb{R}^{O(\log n)}$

- Using J-L:
  - We ensure all $\|\Pi f(u) - \Pi f(v)\|^2 \approx \|f(u) - f(v)\|^2$
  - Go to $O(\log n)$ dimensions, then "take the best coordinate"

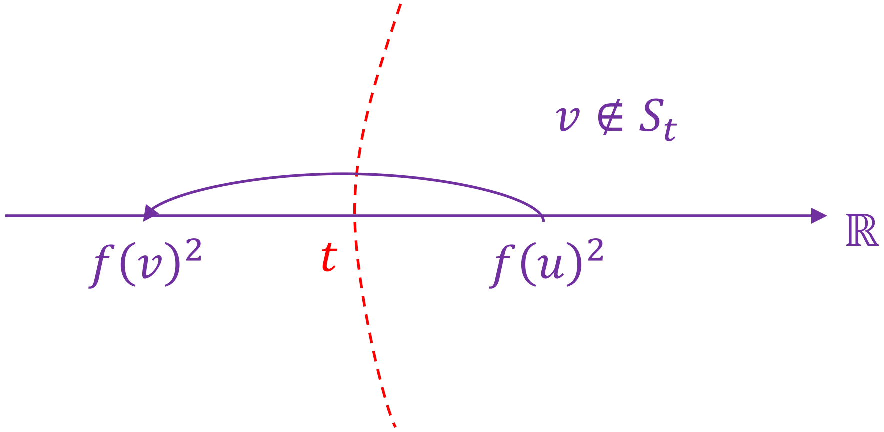
- Using directed program:
  - Objective written as $\sum_u \pi(u) \max_{v:u\to v} \|f(u) - f(v)\|^2$
  - Projected objective: $\sum_u \pi(u) \max_{v:u\to v} (\Pi f(u) - \Pi f(v))^2$
  - Expected max. of d squared standard Gaussians is $O(\log d)$
  - Linearity of expectations to conclude $\gamma^{(1)}(G) \leq O(\log d) \cdot \gamma^{(n)}(G)$

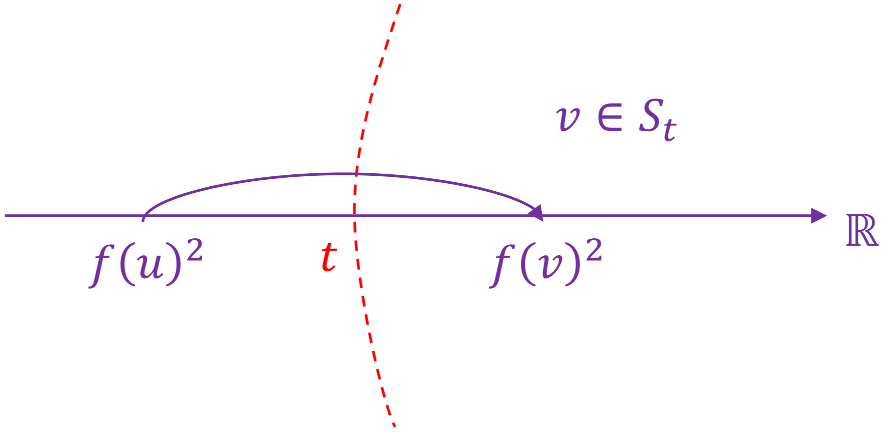
$\Pi: \mathbb{R}^n \to \mathbb{R}$

Threshold rounding



- Take 1D solution $f: V \to \mathbb{R}$
- Consider directed vertex boundary $\vec{\partial}(S)$
- Define $S_t := \{v \in V: f(v)^2 > t\}$





$$\Pr[v \in \vec{\partial}(S_t)] \propto \max_{u:u\to v} |f(u)^2 - f(v)^2|$$

Recap

Reweighted eigenvalue λ_2^*

⬇ Strong duality

Dual program $\gamma^{(n)}(G)$ $\overset{\approx}{\Rightarrow}$ Dir. program $\vec{\gamma}^{(n)}(G)$

⇩ | ⬇ $\log d$ projection

1D dual program $\gamma^{(1)}(G)$ $\overset{\approx}{\Leftrightarrow}$ 1D dir. program $\vec{\gamma}^{(1)}(G)$

⇩ | ⬇ Threshold rounding

Cut $S, \psi(S) \lesssim \sqrt{\lambda_2^* \cdot \log d}$ ⬅ Postprocessing ⬅ Cut $S_t, \overrightarrow{S_t} \lesssim \sqrt{\lambda_2^* \cdot \log d}$

Proxy for λ_k^*

- We don't know how to write λ_k^* as a convex program!
- Idea: consider $\sigma_k^* \coloneqq (\lambda_1 + \lambda_2 + \cdots + \lambda_k)^*$

- Dual looks like

$$\begin{aligned} \kappa(G) := \min_{f:V\to\mathbb{R}^n,\ g:V\to\mathbb{R}_{\geq 0}} \quad & \sum_{v\in V} \pi(v) g(v) \\ \text{subject to} \quad & g(u) + g(v) \geq \|f(u) - f(v)\|^2 \quad \forall uv \in E \\ & \sum_{v\in V} \pi(v) f(v) f(v)^T \preccurlyeq I_n \\ & \sum_{v\in V} \pi(v) \|f(v)\|^2 = k. \end{aligned}$$

Fractional vertex cover

Normalization constraints (but different)

- Convex program! (Factor k loss)

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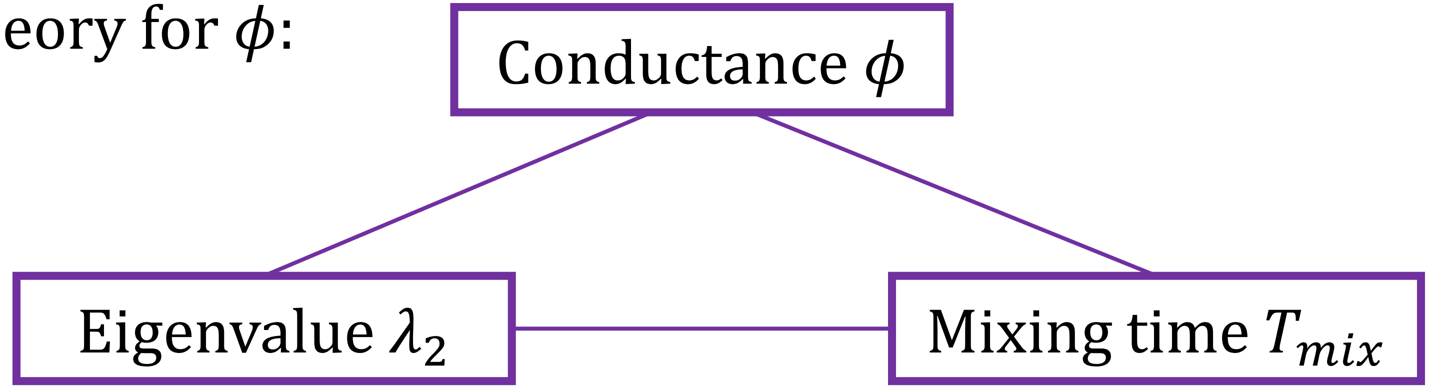
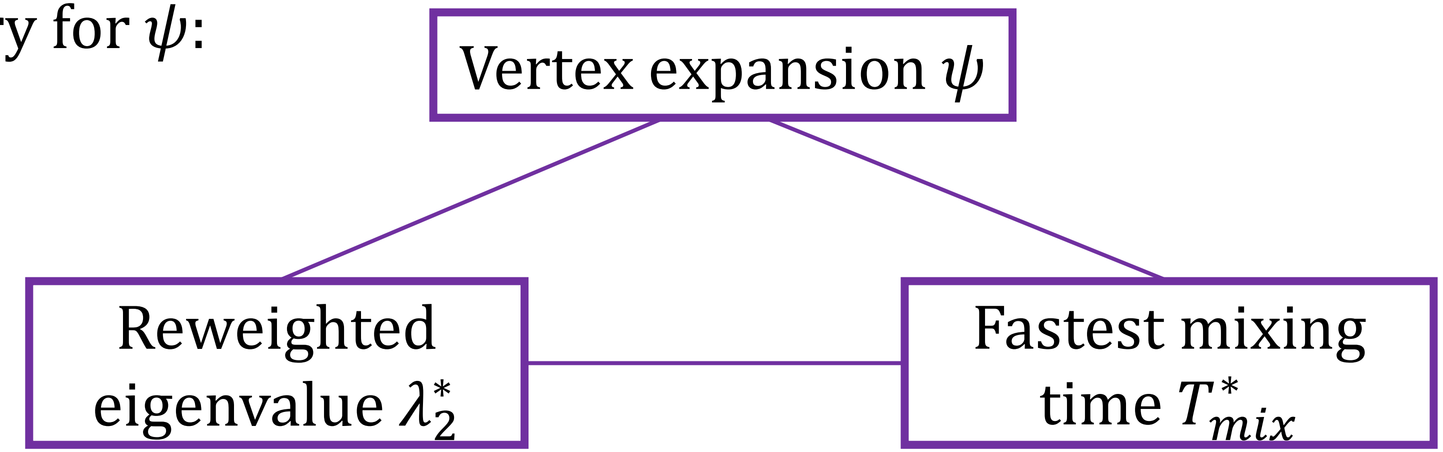
I) Features of new spectral theory

- Classical spectral theory for ϕ:



- New spectral theory for ψ:



II) Cheeger's inequalities

Eigenvalues	Reweighted eigenvalues
Cheeger [Cheeger '70, Alon, Milman '85, Alon '86] Relates λ_2 to ϕ	[Kwok, Lau, T. '22] Relates λ_2^* to ψ
Bipartite Cheeger [Trevisan '09] Relates $2 - \lambda_n$ to "bipartiteness"	[Kwok, Lau, T. '22] Relates "$2 - \lambda_n^*$" to "bipartite v. expn"
Higher-order Cheeger [LOT '12, LRTV '12] Relates λ_k to "k-way conductance"	[Kwok, Lau, T. '22] Relates λ_k^* to "k-way v. expn"
Improved Cheeger [KLLOT '13] Relates λ_2 and λ_k to ϕ	[Kwok, Lau, T. '22] Relates λ_2^* and λ_k^* to ψ

What about reweighted versions of other spectral results?

Open questions

Extension of spectral theory

- Small-set vertex expansion
- Hypergraphs, directed graphs, etc.

Algorithms

- Fast(er) algorithms for approximating ψ or computing λ^*
- Local algorithms for ψ

Reweighting & approximation

- Arora-Ge conjecture for graph coloring
- Steurer's conjecture (true => SUBEXP sparsest cut)

Thank you!