"Interpolation method"

Talk by Ramon van Handel

1° A new proof to Poincaré Inequality

$f : \{-1,1\}^n \longrightarrow \mathbb{R}$

$D_i f(u) = \frac{1}{2}(f(u) - f(u \oplus e_i))$

$e_i = (+1, \dots, +1, \underset{\substack{\uparrow \\ i\text{th} \\ \text{entry}}}{-1}, +1, \dots)$

$u \oplus e_i = (u_1, \dots, u_{i-1}, -u_i, \dots)$

Thm

$\mathrm{Var} f \le \sum_{i \le n} \mathbb{E}[(D_i f)^2]$

If we have a function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}$

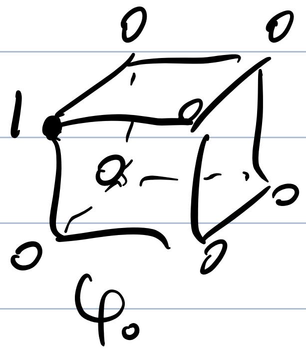
$\gamma(0) = \mathbb{E} f^2$, $\gamma(\infty) = (\mathbb{E} f)^2$

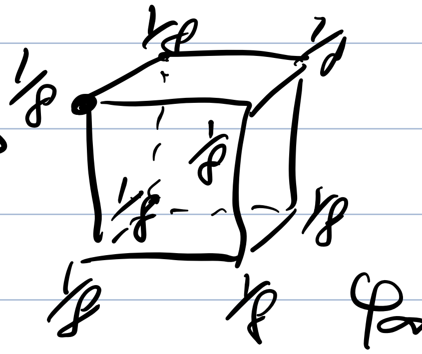
Then,

$\mathrm{Var} f = -\int_0^\infty \gamma'(t)\, dt$.

Define:

$P_t f(u) = \sum_v \varphi_t(v) \cdot f(u \oplus v)$ $\quad \varphi_t(v) = \prod_{i \le n} \left(\frac{1 + v_i \cdot e^{-t}}{2} \right)$





$$Lf(u) = -\sum_{i \le n} D_i f(u)$$

D_i , P_t , L

① $Lf = \lim_{t \to 0^+} \frac{P_t f - f}{t}$, and $\frac{d}{dt}(P_t f) = L P_t f$.

② $P_t = e^{Lt}$

③ $P_{t+s} = P_t \cdot P_s$

④ D_i, P_t, L commute , and $D_i^2 = D_i$.

If we define $\gamma(t) = \mathbb{E}[(P_t f)^2]$, then
$\gamma(0) = \mathbb{E}(f^2)$, $\gamma(\infty) = (\mathbb{E}f)^2$,
and

$$\begin{aligned}
\operatorname{Var} f &= -\int_0^\infty \gamma'(t)\,dt \\
&= -\int_0^\infty \frac{d}{dt}\mathbb{E}[(P_t f)^2]\,dt \\
&= -2\int_0^\infty \mathbb{E}\left[P_t f \cdot \underline{\frac{d}{dt} P_t f}\right] dt \\
&= -2\int_0^\infty \mathbb{E}\left[P_t f \cdot L P_t f\right] dt \\
&= 2\int_0^\infty \mathbb{E}\left[P_t f \cdot \left(\sum_{i \le n} D_i^2\right) P_t f\right] dt
\end{aligned}$$

$L = \sum D_i = -\sum D_i^2$

$$= 2\int_0^\infty \sum_{i \le n} \mathbb{E}[(D_i P_t f)^2]\, dt$$

$\mathbb{E}[f \cdot D_i g]$
$= \mathbb{E}[D_i f \cdot g]$

$(*)$

$$\le 2\int_0^\infty \sum_{i \le n} e^{-2t} \cdot \mathbb{E}[(D_i f)^2]\, dt$$

$$= \sum_{i \le n} \mathbb{E}[(D_i f)^2].$$

$f : \{-1,1\}^n \to \mathbb{R}$

$\hat{f}(u) = \frac{1}{2^n} \sum_v (-1)^{u \oplus v} f(v)$

$\operatorname{Var} f = \sum_{v \ne 1} |\hat{f}(v)|^2$

$\sum_{i \le n} \mathbb{E}[(D_i f)^2] = \sum_v (\#\text{ of } -1\text{'s in } v) \cdot |\hat{f}(v)|^2$

$(*)$: $(D_i P_t f)^2(u) \le e^{-2t} P_t (D_i f)^2 (u)$

$$\varphi_t(v) = \prod_{i \le n} \left(\frac{1 + v_i \cdot e^{-t}}{2} \right)$$

$$\sum_v [\varphi_t(v) + \varphi_t(v \oplus e_i)] (f(u \oplus v) - f(u \oplus v \oplus e_i)) = 0,$$

$$D_i P_t f(u) = \frac{1}{2} \sum_v \varphi_t(v) (f(u \oplus v) - f(u \oplus v \oplus e_i))$$

$$D_i P_t f(u) = \frac{-1}{2} \sum_v \varphi_t(v \oplus e_i) (f(u \oplus v) - f(u \oplus v \oplus e_i))$$

$$\Rightarrow D_i P_t f(u) = \frac{1}{2} \sum_v \left(\frac{\varphi_t(v) - \varphi_t(v \oplus e_i)}{2} \right) \left(f(u \oplus v) - f(u \oplus v \oplus e_i) \right)$$

$$= \frac{1}{2} \sum_v \left[\frac{\varphi_t(v) + \varphi_t(v \oplus e_i)}{2} \cdot \frac{\varphi_t(v) - \varphi_t(v \oplus e_i)}{\varphi_t(v) + \varphi_t(v \oplus e_i)} \right] \cdot \left(f(u \oplus v) - f(u \oplus v \oplus e_i) \right)$$

$$\left(\begin{aligned} \frac{\varphi_t(v)}{\varphi_t(v) + \varphi_t(v \oplus e_i)} &= \frac{\frac{1 + v_i \cdot e^{-t}}{2}}{\frac{1 + e^{-t}}{2} + \frac{1 - e^{-t}}{2}} = \frac{1 + v_i \cdot e^{-t}}{2} \\ \frac{\varphi_t(v \oplus e_i)}{\text{''}} &= \frac{1 - v_i \cdot e^{-t}}{2} \end{aligned} \right)$$

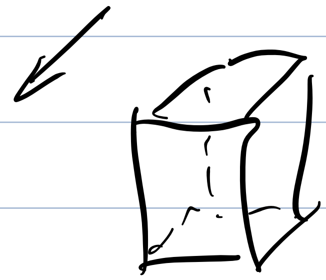
$$= \sum_v \left[\left(v_i \cdot e^{-t} \right) \cdot \frac{\varphi_t(v) + \varphi_t(v \oplus e_i)}{2} \cdot \left(\frac{f(u \oplus v) - f(u \oplus v \oplus e_i)}{2} \right) \right]$$

$$= e^{-t} \sum_v \left[\frac{\varphi_t(v) + \varphi_t(v \oplus e_i)}{2} \right] \cdot v_i \cdot \left(\frac{f(u \oplus v) - f(u \oplus v \oplus e_i)}{2} \right)$$

By Jensen,

$$(D_i P_t f)^2(u) \le e^{-2t} \cdot \sum_v \left[\frac{\varphi_t(v) + \varphi_t(v \oplus e_i)}{2} \right] \cdot \left(\frac{f(u \oplus v) - f(u \oplus v \oplus e_i)}{2} \right)^2$$

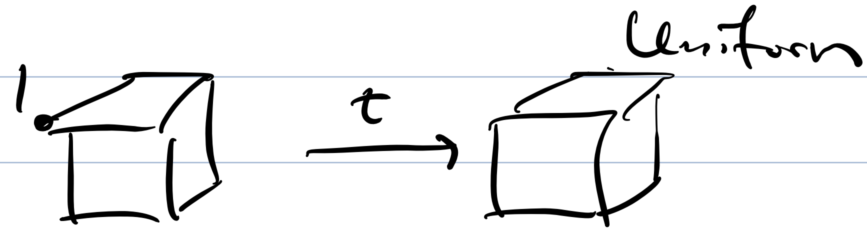
$$= P_t (D_i f)^2 (u).$$



2° Markov semigroup (P_t, L, D_i)

Q: How to come up with $P_t f(u) = \sum_v \varphi_t(v) f(u \oplus v)$.

#1.



- Coordinates are independent & equal

$$\Rightarrow \varphi_t(v) = \prod_{i \le n} \left[\begin{cases} \omega_t & , \text{ if } v_i = 1 \\ 1 - \omega_t & , \text{ if } v_i = -1 \end{cases} \right]$$

- Want the semigroup property to hold
  $(P_{t+s} = P_t \cdot P_s)$

(Get $\omega_{s+t} = \omega_s + \omega_t - 2\omega_s \omega_t$)

$$(1 - 2\omega_{s+t}) = (1 - 2\omega_s)(1 - 2\omega_t)$$

- Using $\{P_t\}$ to define L

$$Lf = \lim_{t \to 0} \frac{1}{t}(P_t f - f)$$

$$L = -\sum D_i \qquad \text{"Infinitesimal generator"}$$

#2. Start from generator L.

One sensible definition would be $L := -\sum_{i \le n} D_i$.

Work out what P_t is by $P_t = e^{Lt}$.

$$P_t = e^{-(\sum_i D_i)t}$$

$$= \prod_{i \le n} e^{-D_i t}$$

$$e^{-D_i t} = \sum_{m \ge 0} \frac{(-D_i t)^m}{m!}$$

$$= I + \sum_{m > 0} D_i \frac{(-t)^m}{m!}$$

$$= I - D_i + \sum_{m \ge 0} D_i \frac{(-t)^m}{m!}$$

$$= I - D_i + e^{-t} D_i$$

$D_i = \frac{I - X_i}{2}$

$$= \left(\frac{1+e^{-t}}{2}\right) I + \left(\frac{1-e^{-t}}{2}\right) X_i$$

X_i swaps u and $u \oplus e_i$.

$$P_t = \prod_{i \le n} \left[\left(\frac{1+e^{-t}}{2}\right) I + \left(\frac{1-e^{-t}}{2}\right) X_i \right]$$

In general, if you have a markov chain (M, P, π),

$L = I - P$

$P_t = e^{-Lt}$

$$\mathcal{E}(f,g) := \sum_u f(u)(Lg)(u)$$

$$\mathcal{E}(f,f) \geq \lambda_2 \cdot \mathrm{Var} f$$

μ_0, $\mu_t = P_t \mu_0$. $\mu_t \to \pi$.

$$f_t(u) := \frac{\mu_t(u)}{\pi(u)}$$

- $\frac{d}{dt} \mathrm{Var}_\pi(f_t) = -2\mathcal{E}(f_t, f_t)$

$$\leq -2\lambda_2 \cdot \mathrm{Var}_\pi(f_t)$$

- $\frac{d}{dt} D(\mu_t \| \pi) = -\mathcal{E}(f_t, \log f_t)$

$$\overset{\text{(MLSI)}}{\leq} -\alpha \cdot D(\mu_t \| \pi)$$

3° Rademacher type & Enflo type.

$$f : \{-1,1\}^n \to \mathbb{R}$$

$$\downarrow$$

$$f : \{-1,1\}^n \to (X, \|\cdot\|_X).$$

Q: Can we still have

$$\mathbb{E}\|f-\mathbb{E}f\|^2 \leq \sum_{i\leq n} \mathbb{E}\|D_i f\|^2 \quad ?$$

$$(X, \|\cdot\|_X) = (\mathbb{R}^n, \|\cdot\|_1),$$

$$f(u) = (u_1, \ldots, u_n)$$

$$\|f(u)\|_1^2 = n^2$$

$$\|D_i f(u)\|_1^2 = 1, \quad \sum_{i\leq n} \mathbb{E}\|D_i f\|_1^2 = n$$

Def (Rademacher type)

Let $p \in [1,2]$. If $\forall x_i \in X$, $f(u) := \sum u_i x_i$, we have

$$\mathbb{E}\|f\|^p \leq C^p \cdot \underbrace{\sum_{j=1}^n \|x_j\|^p}_{\mathbb{E}_u \|D_j f(u)\|^p},$$

then we say that X has "Rademacher" type p.

$T_p^R(X)$: smallest possible const C.

(Enflo type)

Let $p \in [1,2]$. If $\forall f : \{-1,1\}^n \to X$ we have

$$\mathbb{E}_u \left\| \frac{f(u) - f(-u)}{2} \right\|^p \leq C^p \cdot \sum_{j=1}^n \mathbb{E}\|D_j f\|^p,$$

then we say that X has "Enflo" type p.

$T_p^E(X)$: smallest possible const C.

Thm $T_p^R(X) \leq T_p^E(X) \leq \frac{\pi}{\sqrt{2}} T_p^R(X)$

<u>Lemma</u> For $\Phi : X \to \mathbb{R}$ convex, continuous,

$$\mathbb{E}\left[\Phi(f - \mathbb{E}f)\right] \leq \int_0^\infty \mathbb{E}_{\delta, u}\left[\Phi\left(\frac{\pi}{2}\sum_{j=1}^n \delta_j(t) \cdot D_j f(u)\right)\right] \mu(dt)$$

$$\mu(dt) = \frac{2}{\pi} \frac{1}{\sqrt{e^{2t} - 1}} dt$$

$$\xi_j(t) = \begin{cases} 1, & \text{w.p. } \frac{1+e^{-t}}{2} \\ -1, & \text{w.p. } \frac{1-e^{-t}}{2} \end{cases}$$ (independent for different j & t)

$$\delta_j(t) = \frac{\xi_j(t) - \mathbb{E}\xi_j(t)}{\sqrt{\operatorname{Var} \xi_j(t)}} = \frac{\xi_j(t) - e^{-t}}{\sqrt{1 - e^{-2t}}}.$$

<u>Proof.</u> $\Phi[\quad u \quad] = \sup_{u^* \in X^*} \{u^*(u) - \Phi^*(u^*)\}$

$$\mathbb{E}_u\left[\Phi(f(u) - \mathbb{E}f)\right] = \sup_{g^* : \{\pm 1\}^n \to X^*} \left\{\mathbb{E}\left[g^*(u)(f - \mathbb{E}f)\right] - \mathbb{E}_u \Phi^*(g^*(u))\right\}$$

Suffices to prove, $g^* : \{\pm 1\}^n \to X^*$,

$$\mathbb{E}\left[g^*(u)(f - \mathbb{E}f)\right] - \mathbb{E}\left[\Phi^*(g^*)\right] \leq \text{RHS of lemma}$$

$g^*(f)$ identified with $\langle g, f \rangle$ / $\mathbb{E}[g \cdot f]$

$$\mathbb{E}[\langle g, \underline{f - \mathbb{E}f}\rangle] = \mathbb{E}\left[\langle g, -\int_0^\infty \frac{d}{dt}(P_t f)\,dt\rangle\right]$$

Interpolation

$$\gamma_u(0) = f(u),\ \gamma_u(\infty) = \mathbb{E}f$$

$$= \mathbb{E}\left[\langle g, -\int_0^\infty L P_t f\,dt\rangle\right]$$

$$= \sum_{i\le n}\int_0^\infty \mathbb{E}_u\left[\langle D_i P_t g(u), D_i f(u)\rangle\right] dt$$

(*)

$$\int_0^\infty \frac{1}{\sqrt{e^{2t}-1}}\,\mathbb{E}_{\xi,u}\left[\langle g(u\oplus\xi(t)), \sum_{i\le n}\xi_i(t)\cdot D_i f(u)\rangle\right] dt$$

$$= \int_0^\infty \mathbb{E}_{\xi,u}\left[\langle \underline{g(u\oplus\xi(t))}, \underline{\sum_{i\le n}\frac{\pi}{2}\,\xi_i(t)\,D_i f(u)}\rangle\right]\mu(dt)$$

So,

$$\mathbb{E}[g^*(u)(f - \mathbb{E}f)] - \mathbb{E}[\Phi^*(g^*)]$$

$$= \int_0^\infty \mathbb{E}_{\xi,u}\Big[\langle g(u\oplus\xi(t)), \sum_{i\le n}\frac{\pi}{2}\,\xi_i(t)\,D_i f(u)\rangle$$

$$- \Phi^*(g(u\oplus\xi(t)))\Big]\,\mu(dt)$$

$$\leq \int_0^{\infty} \mathbb{E}_{\xi,u}\left[\mathbb{E}\left(\sum_{i\leq n} \frac{\mathbb{I}}{2} \delta_i(t) D_i f(u) \right) \right] \mu(dt)$$

$= RHS$.