Reading Group S21

16 - Formal Hessian as CdV Matrix

Alex Tung

2nd June, 2021

Agenda

- Recall
- Lovasz's CdV construction as volume Hessian
- Volume Hessian is CdV
  - One positive eigenvalue
  - Corank of the matrix
  - The remaining conditions

I. Recall

(Re-)Recall

Problem: Given a graph G, as the 1-skeleton of a polytope $P \subseteq \mathbb{R}^d$, find a CdV matrix of G.

Colin de Verdière

CdV matrix:

1. $M_{ij} < 0$ if $(i,j) \in E$, otherwise $M_{ij} = 0$
2. M has exactly one positive eigenvalue and it is simple.
3. Strong Arnold property

M is an <u>optimal CdV matrix</u> if $\operatorname{Corank}(M)$ is maximized.

Volume Hessian

Suppose polytope $P \subseteq \mathbb{R}^d$ ($\ni \vec{0}$) has vertices $v_1, \ldots, v_n$.

Define for each $x \in \mathbb{R}^n$ the following polytope:

$$P(x) := \{ y \in \mathbb{R}^d : \langle v_i, y \rangle \leq x_i \text{ for all } i \in [n] \}$$

For example, $P(\vec{1})$ is the usual polar of P.

aka Formal Hessian

Volume Hessian of $P(x^0)$ has the form

$$H_{ij} = -\frac{\partial^2 \mathrm{vol}_d(P(x))}{\partial x_i \partial x_j}\bigg|_{x=x^0}$$

Main Theorem: Volume Hessian is CdV

[Thm 2.4] Let $x^\circ \in \mathbb{R}^n_{>0}$ and

$$P(x^\circ) = \{ y \in \mathbb{R}^d : \langle v_i, y \rangle \leq x^\circ_i \text{ for } i \in [n] \}$$

Let G be the dual skeleton of $P(x^\circ)$.

Then,

the volume Hessian $\mathcal{H}$ of $P(x^\circ)$ is a CdV matrix of G, with corank d.

Last Time

Brunn-Minkowski Theorem:

For $A, B \subseteq \mathbb{R}^d$: convex, compact, and $0 \le \lambda \le 1$,

$$\mathrm{vol}\left(\lambda A + (1-\lambda) B\right)^{1/d} \geq \lambda \cdot \mathrm{vol}(A)^{1/d} + (1-\lambda) \cdot \mathrm{vol}(B)^{1/d}.$$

(Proof is by induction on d and a clever parametrization.)

Last Time

Mixed volume (for two bodies):

These are scalars $V(P_{i_1}, \dots, P_{i_d})$ arising from the polynomial expansion of $\mathrm{vol}_d(\lambda_1 P_1 + \dots + \lambda_k P_k)$.

For our setting, useful to know

$$\mathrm{Vol}(\lambda A + (1-\lambda)B) = \sum_{k=0}^{d} \binom{d}{k} \cdot \lambda^k \cdot (1-\lambda)^{d-k} \cdot V(\underbrace{A, \dots, A}_{k}, \underbrace{B, \dots, B}_{d-k}).$$

$$\mathrm{vol}(A + tB) = \sum_{k=0}^{d} \binom{d}{k} \cdot t^{d-k} \cdot V(\underbrace{A, \dots, A}_{k}, \underbrace{B, \dots, B}_{d-k})$$

e.g. $B = B(\vec{0}, 1)$.

Last Time

First and Second Minkowski inequalities

Using the concavity of $f: \lambda \mapsto \mathrm{vol}(\lambda A + (1-\lambda)B)^{1/d}$,
and that $f(0) = f(1) = 0$:

$$f'(0) \geq 0 \longrightarrow V(A, B, \ldots, B)^d \geq V(A) \cdot V(B)^{d-1}$$

$= \mathrm{vol}(B) = V(B, B, \ldots, B)$

$$f''(0) \leq 0 \longrightarrow \boxed{V(A, B, \ldots, B)^2 \geq V(A, A, B, \ldots, B) \cdot V(B)}$$

These are called the 1st and 2nd Minkowski inequalities.

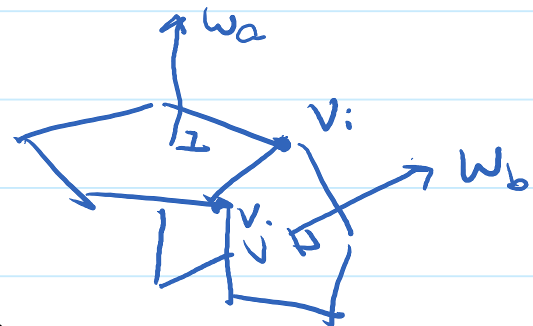
Log-concave results like the second Minkowski inequality is connected to the signature of Hessian.

We will use it to prove that the volume Hessian has one positive eigenvalue (the hardest of the CdV conditions to check).

II. Lovasz's Construction Revisited

Given planar graph G as the 1-skeleton of $P \subseteq \mathbb{R}^3$,
here is how Lovasz constructed an optimal (corank $= 3$) CdV matrix for G.

Take the polar P^* of P with vertices $w_a, w_b, \ldots$



Note that $\langle w_a - w_b, v_i \rangle = 0$, $\langle w_a - w_b, v_j \rangle = 0$.

There exists scalar M_{ij} s.t.

$$\boxed{w_a - w_b = M_{ij} (v_i \times v_j)}.$$

Proved in #14 that M is CdV with corank 3.

This construction is just the volume Hessian in disguise!

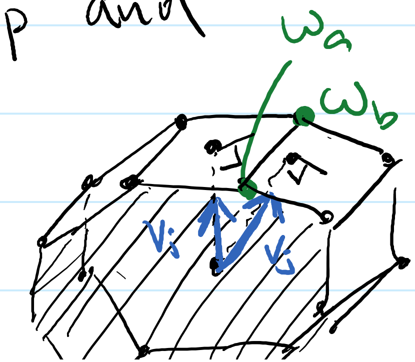
G : 1-skeleton of P

$$P \xrightarrow{\text{polar}} P^* \quad \begin{cases} \mathcal{H}(P^*)\big|_{x=\mathbb{1}} \\ \quad \| \\ M_{ij} = \dfrac{-\|w_a - w_b\|}{\|v_i \times v_j\|} . \end{cases}$$

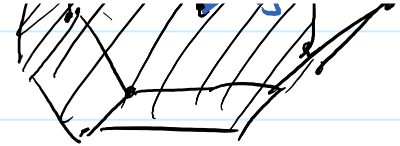
both are nxn matrices!

<u>Lemma</u> In polytope P^* with vertices $w_1, \ldots, w_p$ and polar vertices $v_1, \ldots, v_n$,

$$\frac{\partial^2 \mathrm{vol}(P^*(x))}{\partial x_i \, \partial x_j}\Bigg| = \frac{\|w_a - w_b\|}{\|v_i \times v_j\|}$$



$$\frac{\quad}{\partial x_i \, \partial x_j}\bigg|_{x=1} = \frac{\quad}{\|v_i \times v_j\|}$$



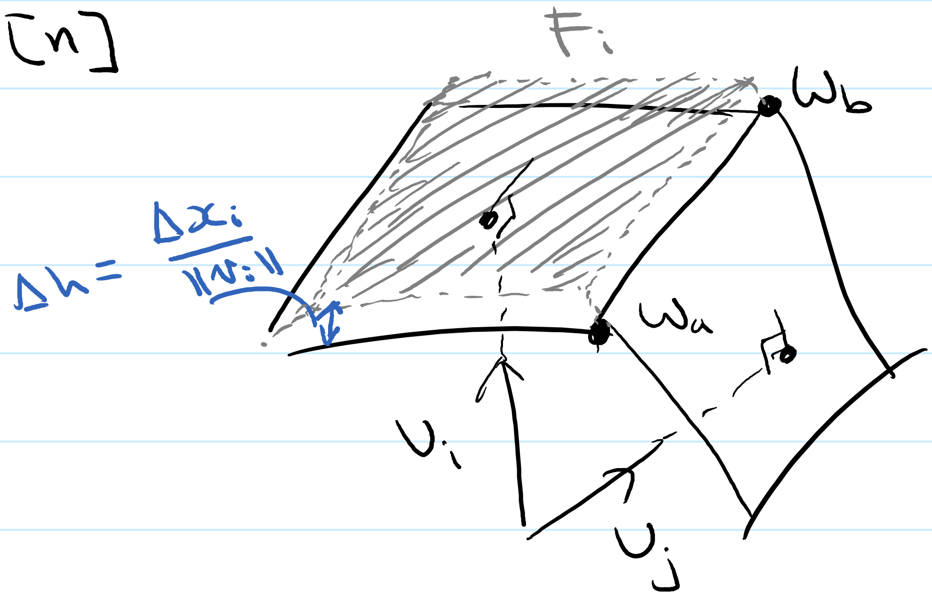
The first derivative

$P^*(x)$ is defined with the constraints

$$\langle v_i , y \rangle \leq x_i \quad , i \in [n]$$

$$\rightsquigarrow \quad \frac{\partial}{\partial x_i} \mathrm{vol}(P^*(x)) \Big|_{x=1}$$

$$= \frac{\mathrm{vol}_2(F_i(x))}{\|v_i\|} \Big|_{x=1}$$



$$\left\langle \frac{v_i}{\|v_i\|} , y \right\rangle \leq \frac{x_i}{\|v_i\|}$$

The second derivative

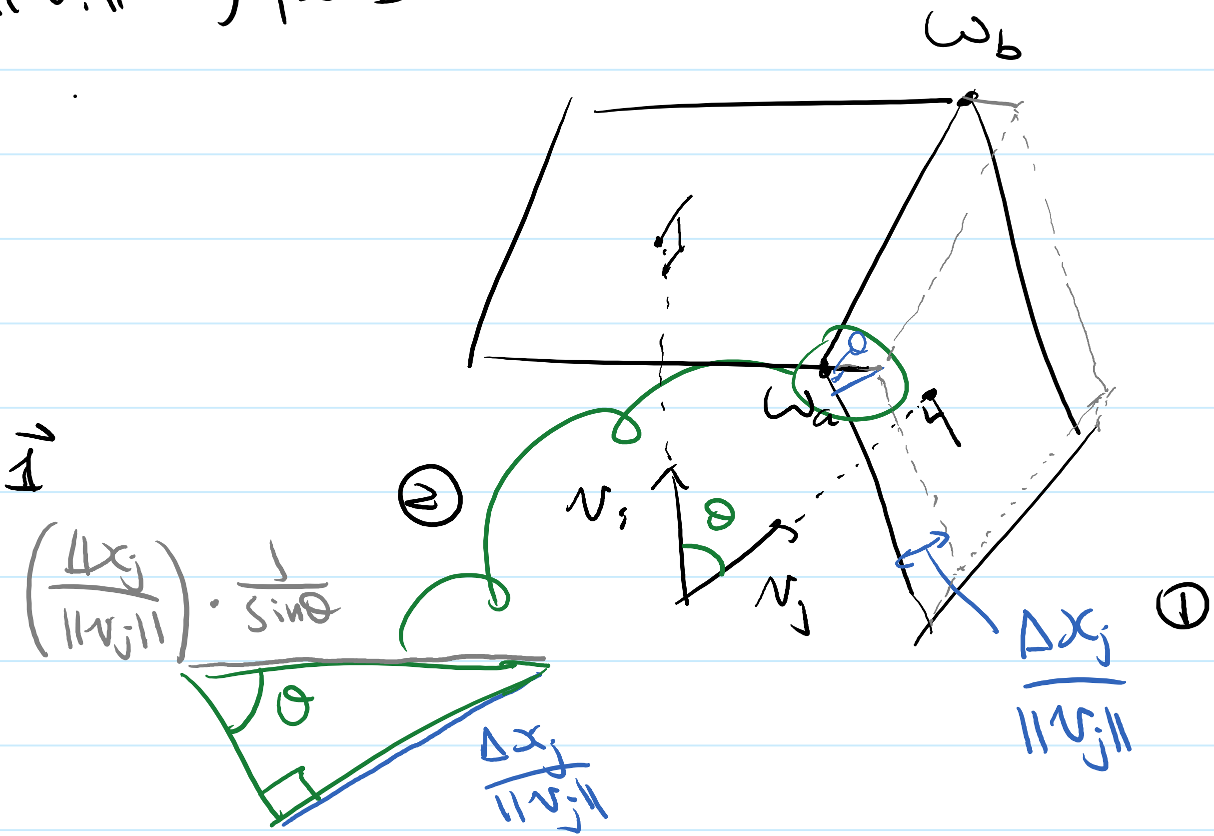
$(i \neq j)$

What is $\frac{\partial}{\partial x_j}\left(\frac{\mathrm{vol}_2(F_i(x))}{\|v_i\|}\right)\Big|_{x=\vec{1}}$?

This is equal to

$$\frac{\mathrm{vol}_1(F_i(x)\cap F_j(x))}{\|v_i\|\cdot\|v_j\|\cdot\sin\theta}\Bigg|_{x=\vec{1}}$$

$$= \frac{\|w_a - w_b\|}{\|v_i \times v_j\|}.$$



θ : angle between v_i and v_j

The diagonal entries

In Lovasz's construction, M_{ii} is chosen so that

$$\sum_j M_{ij} v_j = 0.$$

So we just need to check $\displaystyle\sum_j \left.\frac{\partial^2 \mathrm{vol}(P^*(x))}{\partial x_i \, \partial x_j}\right|_{x=1} \cdot v_j = 0.$

Indeed, by the identity (proved last time under different context)

$$\sum_j \mathrm{vol}_2(F_j) \cdot \frac{v_j}{\|v_j\|} = \vec{0}$$

rewrite $\displaystyle\sum_j \left.\frac{\partial \mathrm{vol}(P^*(x))}{\partial x_j}\right|_{x=1} \cdot v_j = \vec{0} \quad \xrightarrow{\frac{\partial}{\partial x_i}} \quad \sum_j \left.\frac{\partial^2 \mathrm{vol}(P^*(x))}{\partial x_i \, \partial x_j}\right|_{x=1} \cdot v_j = \vec{0}$

G is the 1-skeleton of P $\rightsquigarrow$ G is the <u>dual</u> 1-skeleton of P^*.

Therefore, the following special case the main theorem holds:

Think P^* in Lovasz's construction

Given $Q \subseteq \mathbb{R}^3$ whose dual 1-skeleton is G.
Then,

$$\mathcal{H}(Q) := \left(-\frac{\partial^2 \mathrm{vol}(Q(x))}{\partial x_i \partial x_j} \bigg|_{x=x^0} \right)_{i,j}$$

is a CdV matrix of G, with corank ~~at least~~ 3.

Now let's generalize.

IIIa. One positive eigenvalue

Main Theorem: Volume Hessian is CdV

[Thm 2.4] Let $x \in \mathbb{R}^n_{>0}$ and

$$P(x) = \{ y \in \mathbb{R}^d : \langle v_i, y \rangle \leq x_i \text{ for } i \in [n] \}$$

Let G be the dual skeleton of $P(x)$.

Then,

the volume Hessian $\mathcal{H}$ of $P(x)$ is a CdV matrix of G, with corank d.

Condition 2: M has one (simple) positive eigenvalue

This is where we use second Minkowski's inequality.

$$\mathrm{vol}(X+tY) = V(X) + td \cdot V(X,\dots,X,Y)$$

$$+ t^2 \cdot \frac{d(d-1)}{2} \cdot V(X,\dots,X,Y,Y)$$

$$+ \dots$$

Using shorthand $x \leftrightarrow P(x)$ everywhere,

$$\mathrm{vol}(P(x)+tP(y)) = V(x) + td \cdot V(x,\dots,x,y)$$

$$+ t^2 \cdot \frac{d(d-1)}{2} \cdot V(x,\dots,x,y,y) + \dots$$

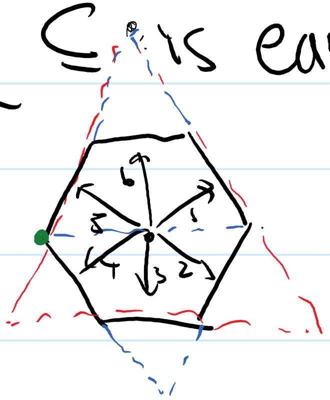
But what we would really like is

$$\begin{pmatrix}1\\2\\1\\2\\1\\2\end{pmatrix}$$

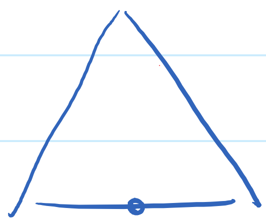
$$\mathrm{Vol}\left(\overset{\color{red}{P(x+ty)}}{\cancel{P(x)+tP(y)}}\right) = V(x) + t\,d\cdot V(x,\dots,x,y) + t^2\cdot\frac{d(d-1)}{2}\cdot V(x,\dots,x,y,y) + \dots$$

Q: When is $P(x) + tP(y) = P(x+ty)$?

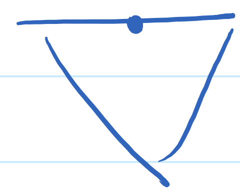
($\subseteq$ is easy to check, but $\supseteq$ may not be true!)

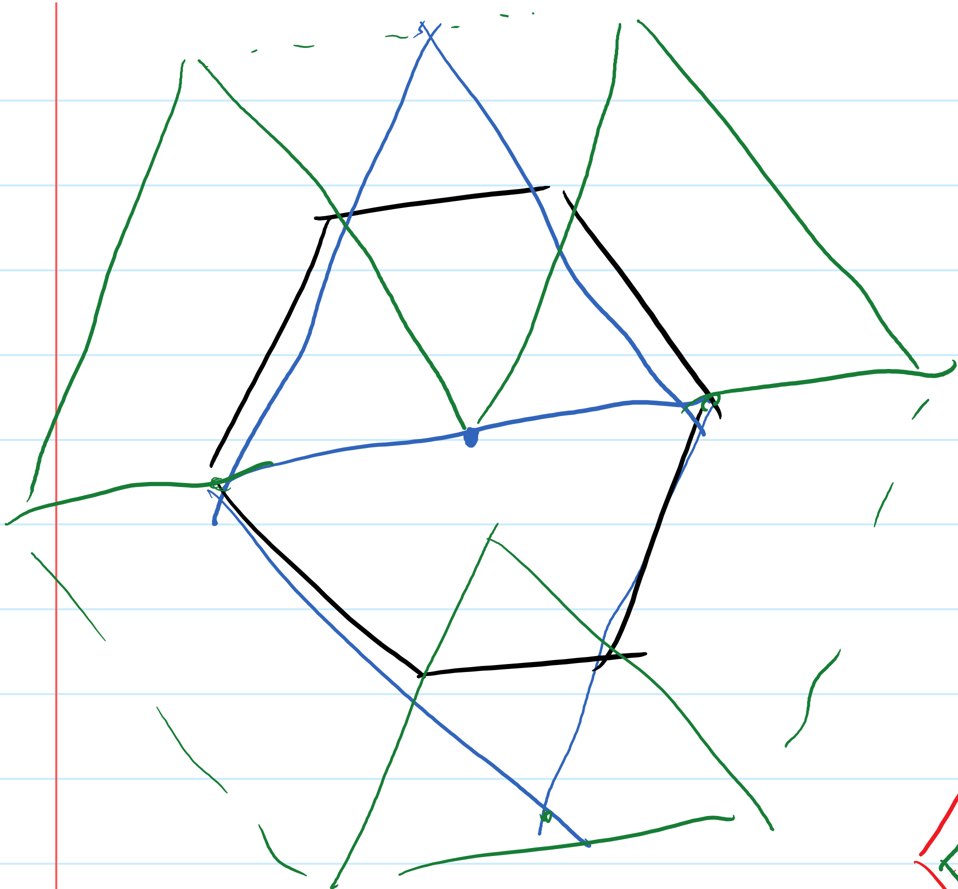


$x = \begin{pmatrix}1\\1\\0\\1\\1\\2 \to 10\end{pmatrix}$



$y = \begin{pmatrix}1\\1\\2 \to 10\\1\\0\end{pmatrix}$

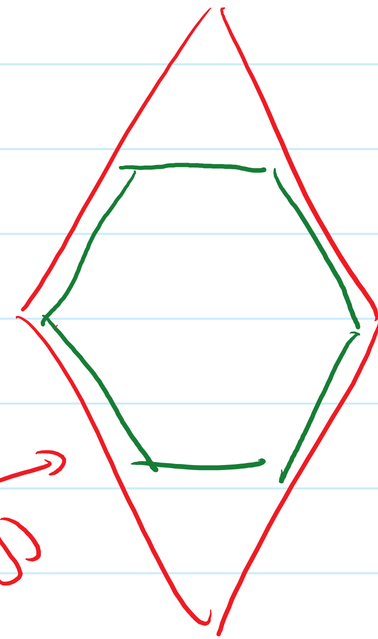


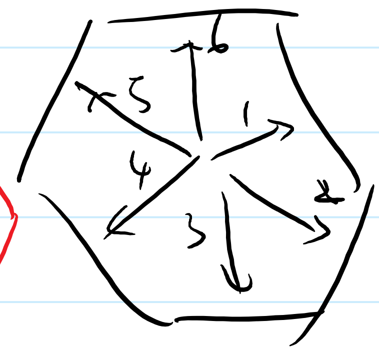


$P(x) + P(y)$

$$\begin{pmatrix} 2 \\ 2 \\ 8 \\ 2 \\ 2 \\ 8 \end{pmatrix}$$

$P(x+y)$





Normal cone

That's why we introduce the definition of normal cone.

Def Given a polytope $P \subseteq \mathbb{R}^d$, and a face F of P, the normal cone of P at F is given by

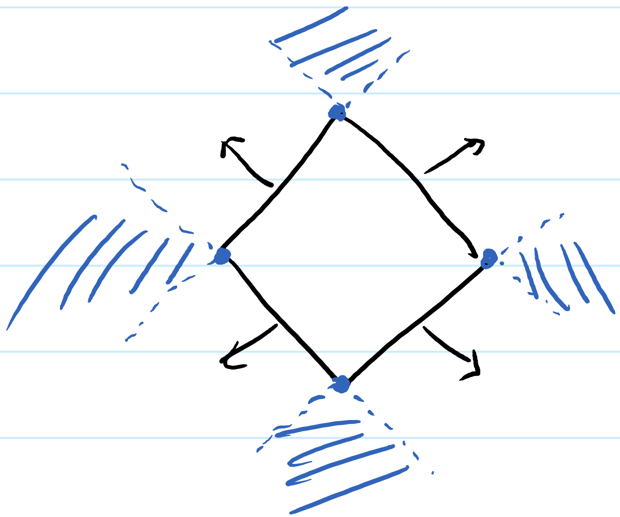
$$N(P, F) := \left\{ u \in \mathbb{R}^d \ : \left(\operatorname*{argmax}_{y \in P} \langle u, y \rangle \right) = F \right\}$$

Normal fan & example

The normal fan of a polytope P is the collection of its normal cones.

Denote it by $N(P)$.

Note that normal cones of P partition $\mathbb{R}^d$.



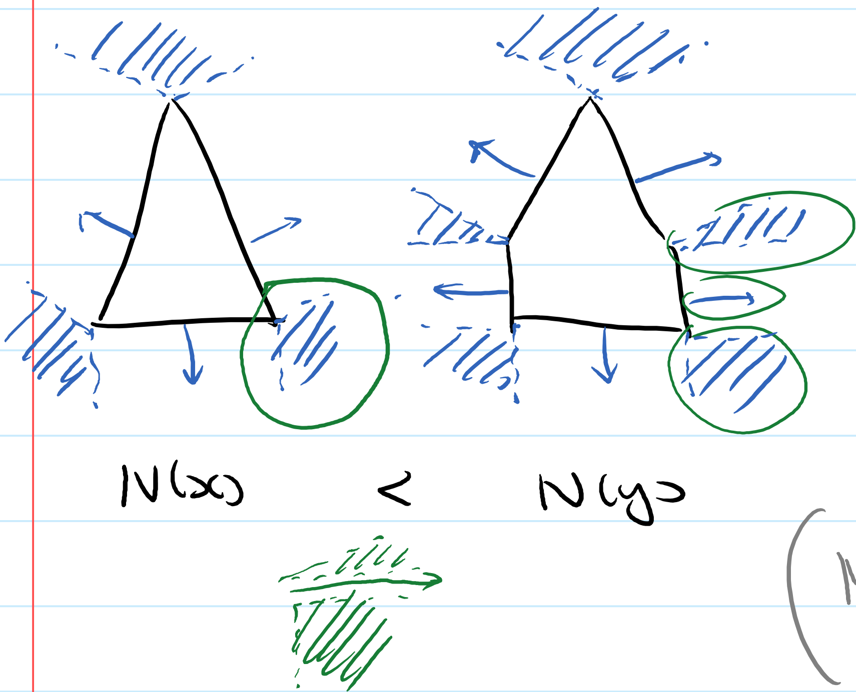
$N(P) = \{ \nwarrow, \nearrow, \swarrow, \searrow,$

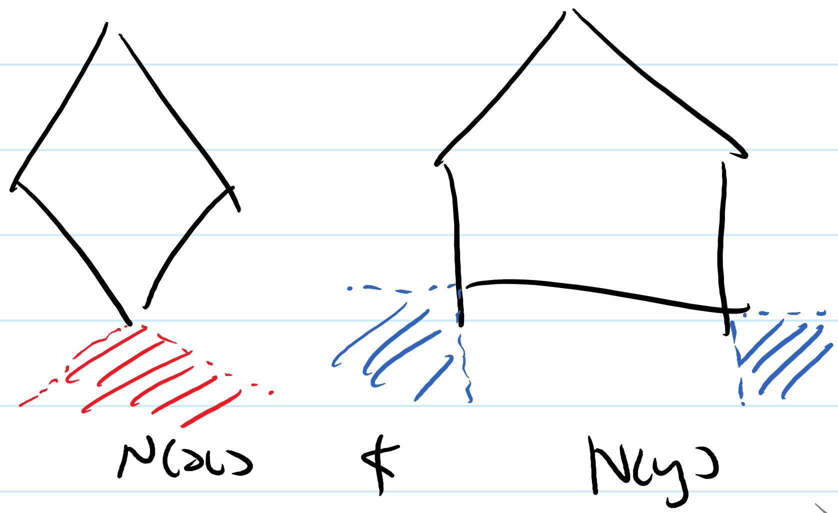
$\}$

Polytopes P(x) and their normal fans

Use $N(x)$ to denote $N(P(x))$, for $x \in \mathbb{R}^n$.

"$N(x) < N(y)$": $N(y)$ subdivides $N(x)$





(Maybe: a constraint that is redundant in $P(y)$ is also redundant in $P(x)$)

Why is subdivision good?

Lemma If $N(x) < N(y)$, then $P(x+y) = P(x) + P(y)$.

Proof ?

$N(x+y) \overset{?}{=} N(y)$

$u \in P(x+y)$.

$\langle u, v_i \rangle \leq x_i + y_i$ for $i \in [n]$

Lemma if $N(y) > N(x)$, then

$$\nabla_y \mathrm{vol}(x) = d \cdot V(y, x, \ldots, x)$$

$$\nabla_y^2 \mathrm{vol}(x) = d(d-1) \cdot V(y, y, x, \ldots, x)$$

Proof. For $t > 0$,

$$\mathrm{vol}(x+ty) = \mathrm{vol}(P(x) + tP(y))$$

$\mathrm{vol}(P(x+ty))$

$$= V(x) + d \cdot t \cdot V(y, x, \ldots, x)$$
$$+ d \cdot (d-1) \cdot \frac{t^2}{2} \cdot V(y, y, x, \ldots, x)$$
$$+ O(t^3)$$

Bilinear form

For $\xi, \eta \in \mathbb{R}^n$, define the following bilinear form

$$\Phi(\xi, \eta) := \nabla_\xi \nabla_\eta \operatorname{vol}(x) .$$

The matrix H is just the representation of Φ in the standard basis.

The bilinear form and mixed volume

By last time's discussion on strongly isomorphic polytopes,

$\mathrm{vol}_d(x)$ is a degree-d homogeneous polynomial in $x_1, \dots, x_n$.

Euler's formula $\rightsquigarrow$ $\nabla_x \mathrm{vol}_d(x) = d \cdot \mathrm{vol}_d(x)$, etc.

$\nabla_x(\mathrm{vol}_d(x)) = \langle x, \nabla \mathrm{vol}_d(x) \rangle$

$$\begin{cases} \Phi(x,x) = \nabla_x \nabla_x \mathrm{vol}_d(x) = d \cdot (d-1) \cdot V(x) \\ \Phi(x,y) = \nabla_x \nabla_y \mathrm{vol}_d(x) = d \cdot (d-1) \cdot V(y, x, \dots, x) \\ \Phi(y,y) = \nabla_y \nabla_y \mathrm{vol}_d(x) = d \cdot (d-1) \cdot V(y, y, x, \dots, x) \end{cases}$$

(by lemma)

f : degree-d homogeneous polynomial in $x_1, \ldots, x_n$.

Then $\langle \bar{x}, \nabla f(\bar{x}) \rangle = d \cdot f(\bar{x})$

The "2-dimension subspace" routine

Lemma Let $L \subseteq \mathbb{R}^n$ be a dim-2 subspace with $x \in L$.
Then $\Phi|_L$ has signature $(+,-)$ or $(+,0)$.

Proof Let's assume $L = \operatorname{span}\{x, y\}$, with $N(y) \supset N(x)$.

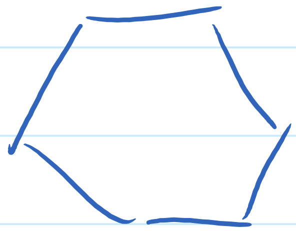
$$\Phi|_L \quad \begin{pmatrix} \Phi(x,x) & \Phi(x,y) \\ \Phi(y,x) & \Phi(y,y) \end{pmatrix}.$$

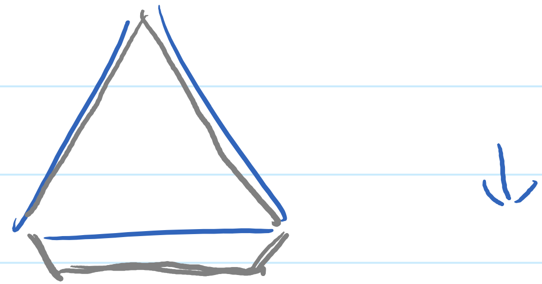
$\operatorname{Det} = \Phi(x,x)\cdot\Phi(y,y) - \Phi(x,y)^2 \leq 0$ by 2^{nd} Minkowski.

Also $\Phi(x,x) = \nabla_x^2 \operatorname{vol}(x) > 0$.
$\operatorname{vol}(x+tx)$

choose

$y = x + \delta$ for some $\delta \in \mathbb{R}^n$. If $|\delta|$ small enough,
"active" constraints in $P(x)$ will not become redundant.
(Relies on the "maybe" condition.)





One positive eigenvalue

Need to take care when y is not necessarily >0.

$$\boxed{\text{Thm The form } \Phi \text{ has exactly one positive eigenvalue.}}$$

Prof $\Phi(x,x)>0 \Rightarrow \Phi$ has at least one +ve eigenvalue

If it has more than one, then $\exists y \in \mathbb{R}^n$, s.t. $\text{span}\{x,y\}$ is a dim. 2 positive definite subspace of $\mathbb{R}^n$.

This contradicts the previous lemma. ∎

Corollary The volume Hessian $\mathcal{H}$ has one (simple) +ve eigenvalue.

IIIb. Corank is d

Second Minkowski and Kernel of Φ

Second Minkowski inequality:

$$V(Q, R, \dots, R)^2 \geq V(R) \cdot V(Q, Q, R, \dots, R).$$

Recall $\Phi\big|_{\mathrm{span}\{x,y\}} = \begin{pmatrix} \Phi(x,x) & \Phi(x,y) \\ \Phi(y,x) & \Phi(y,y) \end{pmatrix}$

has determinant 0 iff

$$V(y, x, \dots, x)^2 = V(x) \cdot V(y, y, x, \dots, x).$$

$\big(Q = P(y), \quad R = P(x)\big)$

Bol's condition

Thm If $\dim(Q) = d$, then

$$V(Q, R, \dots, R)^2 = V(R) \cdot V(Q, Q, \dots, R)$$

iff either
- $\dim(R) < d-1$

or
- "R is homothetic to a $(d-2)$-tangential body of Q"

We won't prove Bol's condition here.

What is an r-tangential body?

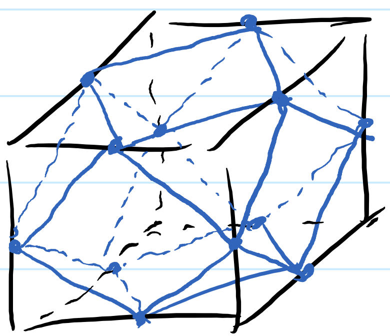
r-tangential Body

Def Given polytopes $K, L \subseteq \mathbb{R}^d$, where $L \supseteq K$,

L is said to be <u>r-tangential to K</u> if

$F \cap K \neq \phi$ for any face F of L of dimension $\geq r$.

<u>Example</u>:

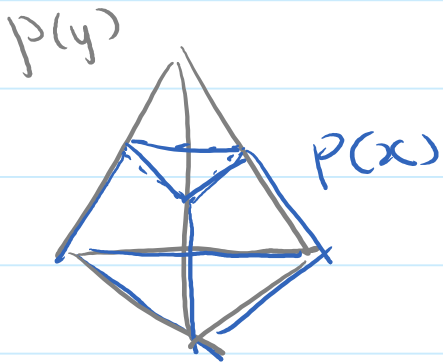


L is 1-tangential to K.

! In Schneider's book the use of "p-extreme support plane" may be confusing.

When are P(x) and P(y) (d-2)-tangential?

? This implies $P(x) = P(y)$?



What if $N(x) < N(y)$?

A, B homothetic

$A = \lambda B + p$, $\lambda > 0$.

"P(x) is homothetic to a (d-2)-tangential body of P(y)"

just means "P(x) is homothetic to P(y)".

Rank of Ker(Φ)

Thm $\operatorname{Ker}(\Phi)$ has rank d.

Proof $\boxed{\text{Step 1}}$: find nullspace of dimension d.

For $p \in \mathbb{R}^d$, define $\bar{p} := \begin{pmatrix} \langle v_1, p \rangle \\ \langle v_2, p \rangle \\ \vdots \\ \langle v_n, p \rangle \end{pmatrix} \in \mathbb{R}^n$.

Claim: $\Phi(\bar{p}, \bar{p}) = 0$.

$P(x + \bar{p}) = p + P(x)$.

$P(x+\bar{p}) = \{ u \in \mathbb{R}^d : \langle u, v_i \rangle \leq x_i + \bar{p}_i \ \forall i \in [n] \}$

$\langle u, v_i \rangle \leq x_i + \bar{p}_i \ \forall i \iff$

$\langle u - p, v_i \rangle \leq x_i \ \forall i \iff u - p \in P(x)$

Proof (cont'd)

$\boxed{\text{Step 2}}$: This is everything in $\mathrm{Ker}(\Phi)$.

Let $\xi \in \mathrm{Ker}(\Phi)$. We need to show that $\xi = \bar{p}$ for some $p \in \mathbb{R}^d$.

Let $L := \mathrm{Span}\{x, \xi\}$.

- $\mathbb{R}\xi = L \cap \mathrm{Ker}(\Phi)$
- Choose $y \in L$ s.t. $\begin{cases} x, y \text{ independent} \\ N(y) > N(x) \end{cases}$.

Bol's condition $\overset{\dots}{\Rightarrow}$ $x = \lambda y + \bar{p}$ for some $\lambda, \bar{p}$.

$\Rightarrow \mathbb{R}\bar{p} = L \cap \mathrm{Ker}(\Phi)$.

$\Rightarrow \xi = \alpha \bar{p}$ for some $\alpha \in \mathbb{R}$.

$\boxtimes$

IIIc. Checking the easy conditions

Condition 1: entries of M

Clearly,

$$H_{ij} := \frac{-\partial \mathrm{vol}_d(P(x))}{\partial x_i \partial x_j}$$

is ≤ 0 when $i \neq j$.

For $i \neq j$:

$(i,j) \in E \iff F_i \cap F_j$ is of "full" dimension $(d-2)$

$\iff \mathrm{vol}_{d-2}(F_i \cap F_j) > 0$

$\iff H_{ij} < 0$.

Condition 3: strong Arnold property

(Strong Arnold property says: if $X \in \mathbb{R}^{n\times n}$ satisfies

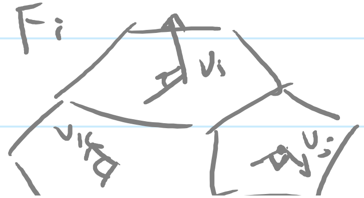
$$\begin{cases} X_{ij} = 0 & \text{if } i=j \text{ or } (i,j)\in E \\ MX = 0 \end{cases}, \quad \text{Then } X = 0.$$
)

Proof.

$$X = \begin{pmatrix} | & | & & | \\ \bar{p}_1 & \bar{p}_2 & \cdots & \bar{p}_n \\ | & | & & | \end{pmatrix} \qquad p_i \in \mathbb{R}^d, \quad \bar{p}_i = \begin{pmatrix} \langle p_i, v_1 \rangle \\ \vdots \\ \langle p_i, v_n \rangle \end{pmatrix}.$$

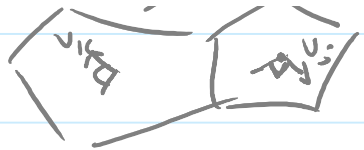
If we fix i. Then $\langle p_i, v_{\underline{j}} \rangle = 0$ for $i=j$ or $(i,j)\in E$.

F_i



$\operatorname{Span}\{v_j : (i,j)\in E \text{ or } j = i\} = \mathbb{R}^d$

$\Rightarrow p_i = 0$ $\qquad \therefore X = 0$



$\Rightarrow p_i = 0$.

$\therefore X = 0$.

Recap

Today we finished off the proof that the volume Hessian is CdV (with corank d).

Discussion

- CdV number and polytope representation
- The unproven geometric facts

$\exists G$ s.t. $CdV(G) > \left(\begin{array}{l} \text{max. dimension of } P \text{ so that} \\ G \text{ is 1-skeleton of } P \end{array} \right)$

$G = K_{\underbrace{2,2,\ldots,2}_{k}}$

$CdV(G) = 2k-3$