Reading Group S21

15 — Volume Hessian Preliminaries

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Agenda

- Recall setting
- CdV matrix as volume Hessian
- Brunn-Minkowski Theorem
- Mixed volumes
- Alexandrov-Fenchel Inequality

Section I: Recall Setting

Recall

Colin de Verdière

① CdV matrix of a graph $G=(V,E)$

M : $n\times n$ symmetric matrix satisfying:

(a) $M_{ij} \begin{cases} <0 & \text{if } (i,j)\in E \\ =0 & \text{if } i\neq j \text{ and } (i,j)\notin E \end{cases}$

(b) M has exactly <u>one positive eigenvalue</u>, which is simple

(c) (Strong Arnold) If X : $n\times n$ matrix s.t. $\begin{cases} X_{ii}=0 \\ X_{ij}=0 \text{ if } (i,j)\in E \end{cases}$

<u>and</u> $XM=0$, then $X=0$.

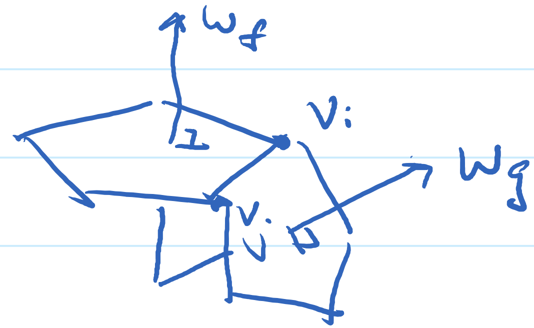
Furthermore, M is <u>optimal</u> if:

(d) Among all matrices satisfying (a)~(c), M <u>maximizes</u> <u>corank</u>.

② Lovasz's construction of CdV matrix

Given graph G as 1-skeleton of polytope $P \subseteq \mathbb{R}^3$.
Vertices are $v_1, \dots, v_n \in \mathbb{R}^3$.

Consider polar P^* of P, with vertices $w_f, \dots$
Say $(i,j) \in E$.



Then $w_f - w_g \perp v_i$ and $w_f - w_g \perp v_j$.

$\rightsquigarrow$ There is scalar M_{ij} s.t.

$$w_f - w_g = M_{ij}(v_i \times v_j).$$

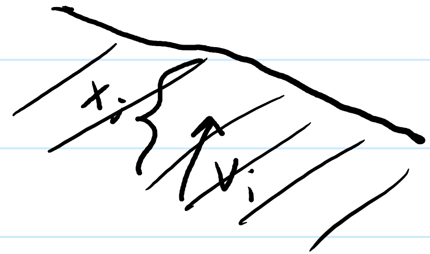
[M_{ii} defined using relation $\sum M_{ij} v_j = \vec{0}$]

Proved last time that M is CdV.

It turns out that Lovasz's Construction can be interpreted as <u>Hessian of volume of polytope</u>!

Define, for $x \in \mathbb{R}^4$,

$$P(x) := \{ y \in \mathbb{R}^3 : \langle v_i, y \rangle \le x_i \}$$



e.g. $P(\vec{1})$ is the usual polar of $P = \mathrm{conv}(\{v_i\})$.

Then, Lovasz's Construction of M satisfies

$$M_{ij} = - \frac{\partial^2 \mathrm{vol}_3(P(x))}{\partial x_i \, \partial x_j} \Bigg|_{x=(1,\dots,1)}.$$

Section II: CdV matrix as volume Hessian

Goal of "Next Week"

Generalize Lovasz's construction:

given G as 1-skeleton of polytope $P \subseteq \mathbb{R}^d$,
Construct a CdV matrix of G.

We will be using the volume Hessian:

$$M_{ij} = - \frac{\partial^2 \mathrm{vol}_d(P(x))}{\partial x_i \, \partial x_j} \Bigg|_{x=(1,\dots,1)} .$$

Key Theorem

[Thm 2.4] Let $x^0 \in \mathbb{R}^n_{>0}$ and

$$P(x^0) = \{p \in \mathbb{R}^d \mid \langle v_i, p\rangle \leq x^0_i \text{ for } i \in [n]\}.$$

Let G be the dual 1-skeleton of $P(x^0)$. Then:

- The matrix M

$$M_{ij} = \left. \frac{-\partial^2 \mathrm{vol}_d P(x)}{\partial x_i \partial x_j} \right|_{x = x^0}$$

is a CdV matrix of the graph G

- The corank of M is equal to d.

Goal of today

Prepare for next week :D

For now, let's forget about the setting and focus on the volume Hessian

$$\left(- \frac{\partial^2 \mathrm{vol}_d(P(x))}{\partial x_i \, \partial x_j} \right)_{i,j} .$$

To understand its properties, we introduce:

$$\begin{cases} \text{Brunn-Minkowski Theorem} \\ \text{Mixed Volumes} \\ \text{Alexandrov-Fenchel Inequality} \\ \text{Bol's condition} \rightsquigarrow \text{Next time.} \end{cases}$$

Section III : Brunn-Minkowski Theorem

Brunn–Minkowski Theorem

Statement: Let $A, B \subseteq \mathbb{R}^d$ be convex, compact. Then, for $\lambda \in [0,1]$,

$$\mathrm{vol}_d\left(\lambda A + (1-\lambda)B\right)^{1/d} \geq \lambda \cdot \mathrm{vol}_d(A)^{1/d} + (1-\lambda)\,\mathrm{vol}_d(B)^{1/d}.$$

In other words, the function $X \mapsto \mathrm{vol}_d(X)^{1/d}$ is concave.

X, Y, then

$$X + Y := \{x + y : x \in X, y \in Y\} \quad \text{(Minkowski sum)}$$

$$\lambda X := \{\lambda x : x \in X\}.$$

$$\boxed{\mathrm{vol}_d(\lambda A + (1-\lambda)B)^{1/d} \geq \lambda \cdot \mathrm{vol}_d(A)^{1/d} + (1-\lambda)\,\mathrm{vol}_d(B)^{1/d}}$$

Proof.

Step 1: deal with boundary cases

$\dim(A)$, $\dim(B) < d \Rightarrow \mathrm{RHS} = 0$.

$\dim(A) < d$, $\dim(B) = d$.

$x \in A$. $\lambda x + (1-\lambda)B \subseteq \lambda A + (1-\lambda)B$.

$$\mathrm{LHS} = \mathrm{vol}_d(\lambda A + (1-\lambda)B)^{1/d} \geq \mathrm{vol}_d(\lambda x + (1-\lambda)B)^{1/d}.$$

$$= (1-\lambda)\cdot \mathrm{vol}_d(B)^{1/d} . = \mathrm{RHS}$$

$$\boxed{\mathrm{vol}_d\left(\lambda A+(1-\lambda)B\right)^{1/d} \geq \lambda\cdot \mathrm{vol}_d(A)^{1/d} + (1-\lambda)\,\mathrm{vol}_d(B)^{1/d}}$$

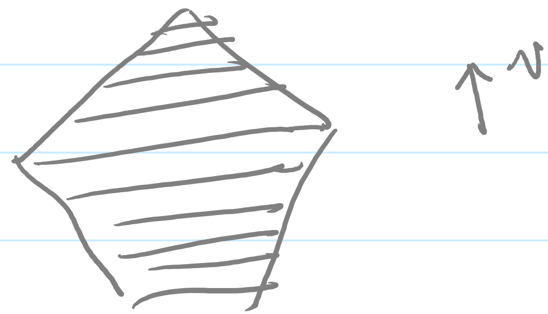
Step 2: Set up induction.

$d=1$: straightforward to check.

Fix $v \in \mathbb{R}^d \setminus \{0\}$, $\|v\|=1$.

$C := \lambda A + (1-\lambda)B$.

$H_C(x) := \{ y \in C : \langle v, y\rangle = x \}$

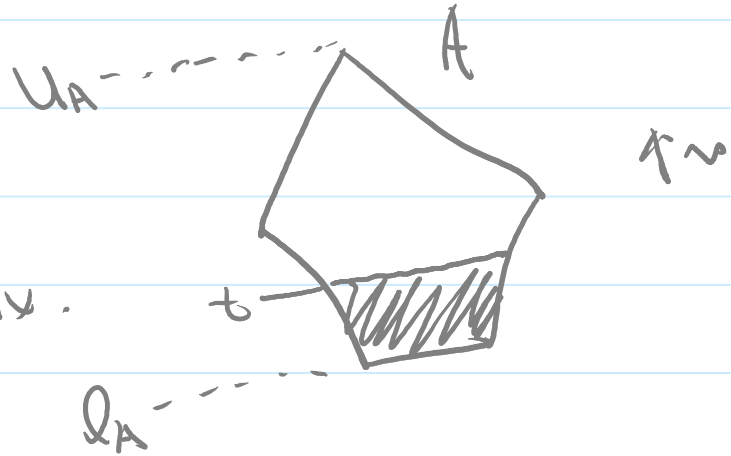


$$\mathrm{vol}_d(C) = \int_{l_C}^{u_C} \mathrm{vol}_{d-1}(H_C(x))\,dx$$

How to parametrize?

$w_A : [l_A, u_A] \to [0,1]$.

$t \mapsto \int_{l_A}^{t} \mathrm{vol}_{d-1}(H_A(x))\,dx$.



w_B defined similarly.

$z_A : [0,1] \to [l_A, u_A]$, inverse of w_A.

$z_B : [0,1] \to [l_B, u_B]$, inverse of w_B.

$z_C := \lambda \cdot z_A + (1-\lambda)\cdot z_B$.

$$\mathrm{vol}_d(C) = \int_{l_C}^{u_C} \mathrm{vol}_{d-1}(H_C(x))\,dx$$

$$= \int_0^1 \mathrm{vol}_{d-1}(\underline{H_C(z_C(t))}) \cdot \underline{z_C'(t)}\,dt.$$

$$z_A'(t) = \frac{1}{w_A'(z_A(t))} = \frac{1}{\mathrm{vol}_{d-1}(H_A(z_A(t)))}$$

$$z_B'(t) = \frac{1}{\mathrm{vol}_{d-1}(H_B(z_B(t)))}$$

$$\boxed{z_C'(t) = \frac{\lambda}{\mathrm{vol}_{d-1}(H_A(z_A(t)))} + \frac{1-\lambda}{\mathrm{vol}_{d-1}(H_B(z_B(t)))}}$$

$$\mathrm{vol}_{d-1}\left(H_C(z_C(t))\right)$$

$$z_C(t) = \lambda \cdot z_A(t) + (1-\lambda)\cdot z_B(t).$$

$$H_C(z_C(t)) \supseteq \lambda \cdot \underset{\substack{\cup \\ a}}{H_A(z_A(t))} + (1-\lambda)\cdot \underset{\substack{\cup \\ b}}{H_B(z_B(t))}$$

- $a+b \in C$
- $\langle a, v\rangle = \lambda \cdot z_A(t)$, $\langle b, v\rangle = (1-\lambda)\cdot z_B(t)$

$\Rightarrow \langle a+b, v\rangle = z_C(t)$.

$$\mathrm{vol}_{d-1}(H_C(z_C(t)))$$

$$\geq \mathrm{vol}_{d-1}\left(\lambda \cdot H_A(z_A(t)) + (1-\lambda)\cdot H_B(z_B(t))\right)$$

$$\geq \left[\lambda \cdot \underbrace{\mathrm{vol}_{d-1}(H_A(z_A(t)))}_{\alpha(t)}^{\frac{1}{d-1}} + (1-\lambda)\cdot \underbrace{\mathrm{vol}_{d-1}(H_B(z_B(t)))}_{\beta(t)}^{\frac{1}{d-1}}\right]^{d-1}$$

$$\mathrm{vol}_d(C) \geq$$

$$\int_0^1 \left[\lambda \cdot \alpha(t)^{\frac{1}{d-1}} + (1-\lambda)\cdot \beta(t)^{\frac{1}{d-1}}\right]^{d-1} \cdot \left[\frac{\lambda}{\alpha(t)} + \frac{1-\lambda}{\beta(t)}\right] dt$$

Given $\lambda \in (0,1)$, $\alpha, \beta > 0$, $p \geq 1$,

$$\left[\lambda \cdot \alpha^{1/p} + (1-\lambda) \cdot \beta^{1/p}\right]^p \cdot \left(\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}\right) \geq 1 .$$

$$\left[\lambda \cdot \alpha^{1/p} + (1-\lambda) \cdot \beta^{1/p}\right]^p \geq \alpha^{\lambda} \cdot \beta^{1-\lambda} \geq \frac{1}{\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}} .$$

① $\lambda \cdot (\alpha^{1/p}) + (1-\lambda) \cdot (\beta^{1/p}) \geq (\alpha^{1/p})^{\lambda} \cdot (\beta^{1/p})^{1-\lambda}$.

② $\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta} \geq \left(\frac{1}{\alpha}\right)^{\lambda} \cdot \left(\frac{1}{\beta}\right)^{1-\lambda}$.

Equality conditions :

① $A+B$ is contained in a hyperplane.

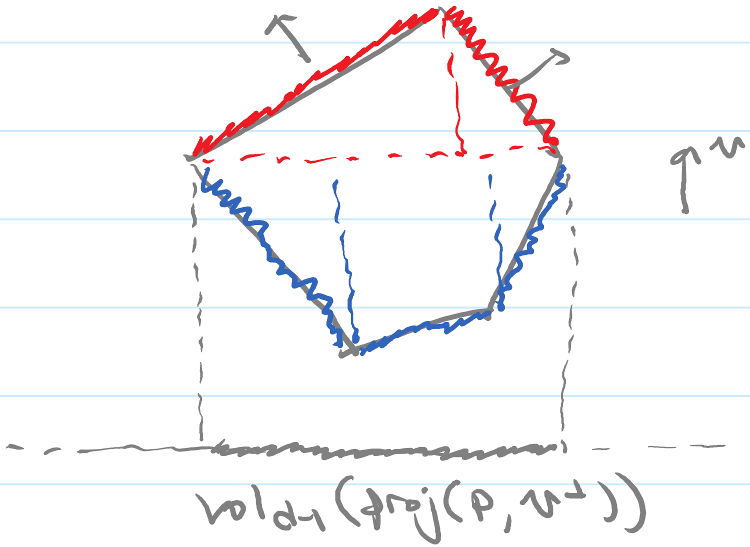
② A, B are homothetic.
(scaling, translation)

Section IV : Mixed Volumes

Lemma Let $P \subseteq \mathbb{R}^d$ be a polytope with facets $F_1, \dots, F_N$ and outward-pointing normals $u_1, \dots, u_N$, each u_i of unit length.

Then,
$$\sum_{i=1}^{N} \mathrm{vol}_{d-1}(F_i) \cdot u_i = \vec{0}.$$

Proof. Choose $v \in \mathbb{R}^d \setminus \{0\}$, $\|v\| = 1$.



$$\mathrm{vol}_{d-1}(\mathrm{proj}(P, v^\perp))$$
$$= \sum_{\langle u_i, v\rangle \geq 0} \mathrm{vol}_{d-1}(F_i) \cdot \langle u_i, v\rangle$$
$$= -\sum_{\langle u_i, v\rangle < 0} \mathrm{vol}_{d-1}(F_i) \cdot \langle u_i, v\rangle$$

$$\frac{1}{d}\sum_{i=1}^{N} h(P+x, u_i)\cdot \mathrm{vol}_{d-1}(F_i)$$

$$= \frac{1}{d}\sum_{i=1}^{N} h(P,u_i)\cdot \mathrm{vol}_{d-1}(F_i) + \frac{1}{d}\sum_{i=1}^{N} \langle x,u_i\rangle \cdot \mathrm{vol}_{d-1}(F_i)$$

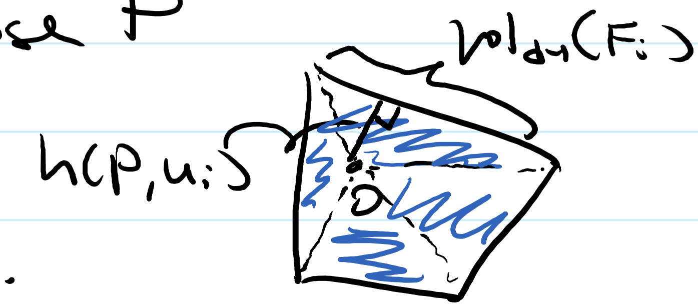
Corollary

$$\mathrm{Vol}_d(P) = \frac{1}{d}\sum_{i=1}^{N} h(P,u_i)\cdot \mathrm{Vol}_{d-1}(F_i)$$

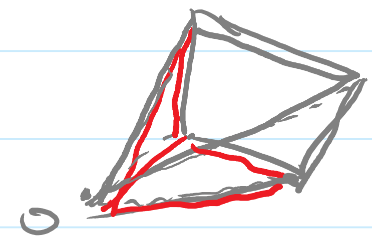
$$h(P,u_i) := \max_{x\in P} \langle x,u_i\rangle .$$

Proof. Due to the lemma, RHS is translation invariant.

Translate so that $\vec{0}\in P^\circ$. Decompose P into pyramids and use



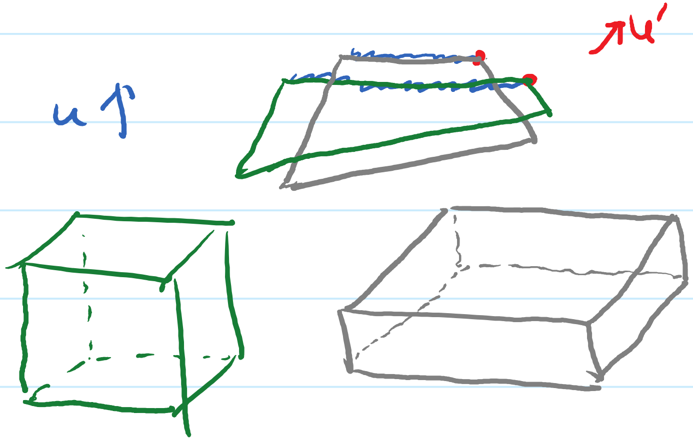
$$\mathrm{vol}(\text{Pyramid}) = \frac{1}{d}\cdot \text{base}\cdot\text{height} .$$



a-type and strongly isomorphic polytopes

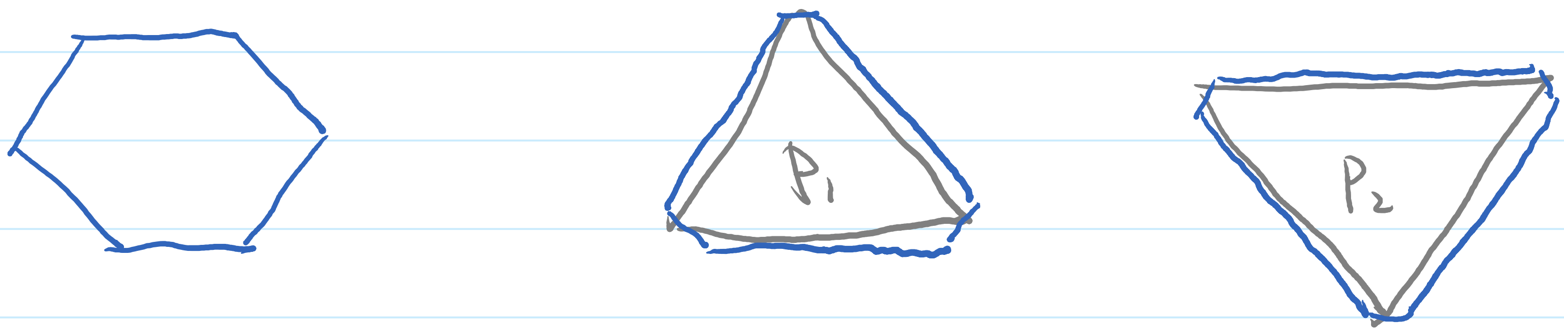
Def. Two polytopes P, Q are strongly isomorphic if, for any direction u,

$$\dim\left(\operatorname*{argmax}_{x \in P} \langle x, u \rangle\right) = \dim\left(\operatorname*{argmax}_{y \in Q} \langle y, u \rangle\right).$$

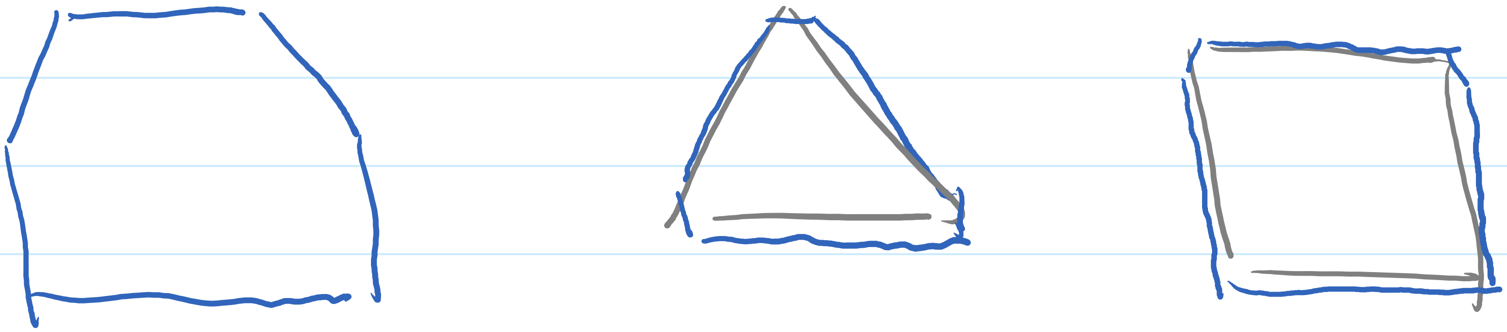


Def. An a-type is an equivalence class of strongly isomorphic polytopes.

<u>Fact</u>. Any finite collection of polytopes $P_1, \ldots, P_N$ can be approximated by simple, strongly isomorphic polytopes $Q_1, \ldots, Q_N$ to arbitrary fineness.



[For details, refer to Section 2.4 of Schneider's book.]



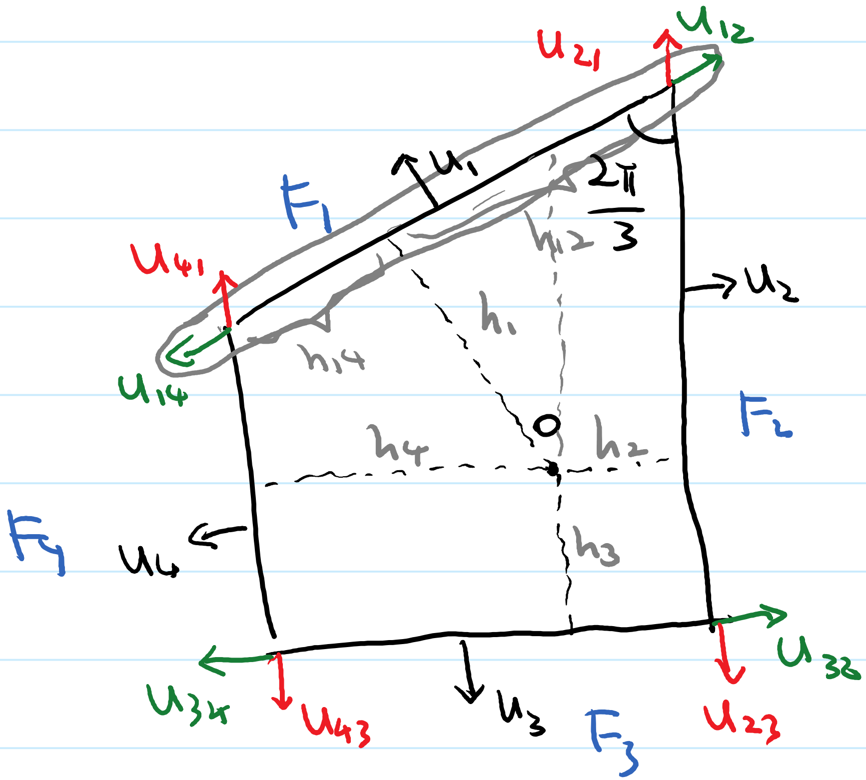
Volume in terms of height parameters

Lemma Given an α-type $\mathcal{A}$, the volume of $P \in \mathcal{A}$ can be represented as

$$\mathrm{vol}_d(P) = \sum a_{j_1,\dots,j_d} h_{j_1} \cdots h_{j_d},$$

such that:

- $j_1, \dots, j_d$ sum over $\{1, \dots, N\}$ (represents the facets)
- $h_i := \max\limits_{x \in P} \langle u_i, x \rangle$, the "height" of P along direction u_i.
- The coefficients $a_{j_1,\dots,j_d}$ are symmetric and depend only on $\mathcal{A}$.

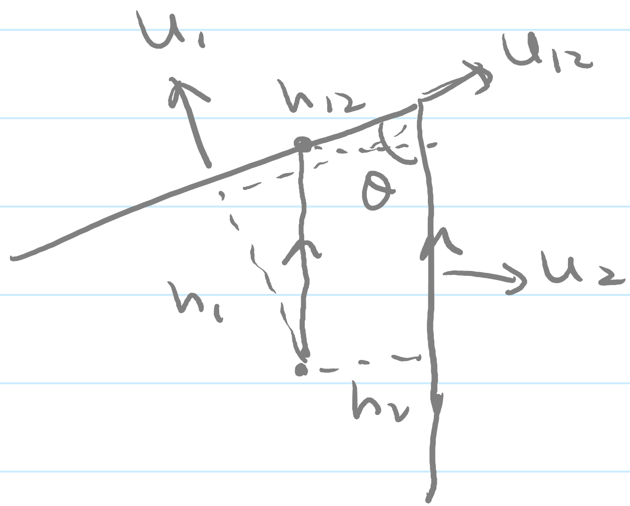


$$\mathrm{vol}(P) = \frac{1}{2}\big(h_1 \cdot \mathrm{vol}(F_1) + h_2 \cdot \mathrm{vol}(F_2)$$
$$+ h_3 \cdot \mathrm{vol}(F_3) + h_4 \cdot \mathrm{vol}(F_4)\big)$$

$$\mathrm{vol}(F_1) = h_{12} + h_{14}$$

$$h_{12} = \frac{1}{\sqrt{3}} h_1 + \frac{2}{\sqrt{3}} h_2$$

$$h_{14} = \frac{-1}{\sqrt{3}} h_1 + \frac{2}{\sqrt{3}} h_4$$



$$h_{12} = h_1 \cdot \cot\theta + h_2 \cdot \csc\theta .$$

$$U_{12} \begin{cases} \perp u_1 \\ \perp F_{12} \\ \text{pointing outward from } F_1 \end{cases}$$

In general:

$$\mathrm{vol}_d(P) = \frac{1}{d}\sum_{i=1}^{N} h_i \cdot \overbrace{\mathrm{vol}_{d-1}(F_i)}^{\in \mathbb{R}_{d-1}[h_{ij}]}.$$

Induction hypothesis + substituting h_{ij} by linear combination of h_i:

$\rightsquigarrow$ $\mathrm{vol}_d(P)$ is also of the form

$$\sum a_{j_1,\dots,j_d} h_{j_1} \cdots h_{j_d}.$$

If A is fixed, then $a_{j_1,\dots,j_d}$ are fixed.

Mixed volume for strongly isomorphic polytopes.

<u>Def</u> Let $P_1, \dots, P_d \in \mathcal{A}$.

Use $F_i^{(k)}$ to denote the i-th facet of P_k;

$h_i^{(k)}$ to denote height of P_k along u_i.

Define

$$V(P_1, \dots, P_d) := \sum_{j_1, \dots, j_d} a_{j_1, \dots, j_d} h_{j_1}^{(1)} \cdots h_{j_d}^{(d)}.$$

Minkowski sum and Mixed volume

Let $P = \lambda_1 P_1 + \cdots + \lambda_n P_n$, where $P_1, \ldots, P_n \in \mathcal{A}$, $\lambda_i \geq 0$.
(Fact: $P \in \mathcal{A}$ as well)

$h(P_i, u_j)$

From
$$\begin{cases} h(P, u_j) = \sum \lambda_i \cdot h_j^{(i)} \\ \mathrm{Vol}_d(P) = \sum a_{j_1,\ldots,j_d} h(P, u_{j_1}) \cdots h(P, u_{j_d}) \\ V(P_1, \ldots, P_d) := \sum_{j_1,\ldots,j_d} a_{j_1,\ldots,j_d} h_{j_1}^{(1)} \cdots h_{j_d}^{(d)} . \end{cases}$$

we get:
$$\mathrm{Vol}_d(P) = \sum_{i_1,\ldots,i_d=1}^{n} \lambda_{i_1} \cdots \lambda_{i_d} V(P_{i_1}, \ldots, P_{i_d}) .$$

$$\boxed{\mathrm{Vol}_d(P) = \sum_{i_1,\dots,i_d=1}^{n} \lambda_{i_1}\cdots\lambda_{i_d} V(P_{i_1},\dots,P_{i_d})}.$$

Significance:

- $\mathrm{Vol}_d(\lambda_1 P_1 + \dots + \lambda_n P_n)$ regarded as a homogeneous degree-d polynomial in $\lambda_1,\dots,\lambda_n$
- The mixed volumes $V(P_{i_1},\dots,P_{i_d})$ are coefficients of this polynomial.

By approximating $P_1,\dots,P_n$ with strongly isomorphic polytopes, the formula extends to general polytopes as well.

Moreover, V is linear under Minkowski sum:

$$V(X,P_{i_2},\dots,P_{i_d}) + V(Y,P_{i_2},\dots,P_{i_d}) = V(X+Y,P_{i_2},\dots,P_{i_d})$$

[Refer to section 5.1 of Schneider's book.]

Section $\underline{\overline{V}}$: Alexandrov-Fenchel Inequality.

Mixed volume:

$$\mathrm{vol}_d(P) = \sum_{i_1,\dots,i_d=1}^{n} \lambda_{i_1}\cdots\lambda_{i_d} V(P_{i_1},\dots,P_{i_d}).$$

Alexandrov-Fenchel Inequality is a "log-concave" type of inequality relating the mixed volumes.

<u>Thm</u> For convex bodies $P_1,\dots,P_d$,

$$V(\underline{P_1},\underline{P_2},P_3,\dots,P_d)^2 \geq V(\underline{P_1},\underline{P_1},P_3,\dots,P_d)\cdot V(\underline{P_2},\underline{P_2},P_3,\dots,P_d).$$

We will only prove a special case

<u>Thm</u> (Minkowski's Inequalities)

For $A, B \subseteq \mathbb{R}^d$: convex bodies of dimension d,

(i) $$V(A, B, \ldots, B)^d \geq \underbrace{V(A)}_{V(A,\ldots,A) = \mathrm{vol}_d(A)} \cdot V(B)^{d-1}$$

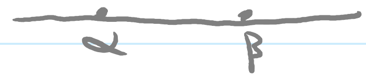
(ii) $$V(A, B, \underbrace{B, \ldots, B}_{d-2})^2 \geq V(A, A, B, \ldots, B) \cdot V(B, B, \ldots, B)$$

Lemma The function

$$f(\lambda): \lambda \mapsto \mathrm{vol}_d(\lambda A + (1-\lambda)B)^{1/d}$$

is concave on $[0,1]$.

$f(\lambda\alpha + (1-\lambda)\beta) \geq \lambda f(\alpha) + (1-\lambda)\cdot f(\beta)$.



Proof. Let $\alpha, \beta \in [0,1]$, $\lambda \in [0,1]$

$$\mathrm{LHS} = \mathrm{vol}_d\left[(\lambda\alpha + (1-\lambda)\beta)A + (1-\lambda\alpha - (1-\lambda)\beta)B\right]^{1/d}$$

$$\mathrm{RHS} = \lambda \cdot \mathrm{vol}_d\left[\alpha A + (1-\alpha)B\right]^{1/d} + (1-\lambda)\cdot \mathrm{vol}_d\left[\beta A + (1-\beta)B\right]^{1/d}$$

$$\lambda \cdot (\alpha A + (1-\alpha)B) + (1-\lambda)\cdot(\beta A + (1-\beta)B).$$

So LHS $\geq$ RHS by Brunn-Minkowski.

The function $f(\lambda) := \mathrm{vol}_d(\lambda A + (1-\lambda)B)^{1/d}$
$- \lambda \cdot \mathrm{vol}_d(A)^{1/d} - (1-\lambda)\cdot \mathrm{vol}_d(B)^{1/d}$

has the following properties:

- f is concave on $[0,1]$.
- $f(0) = f(1) = 0$.
- $f \geq 0$ on $[0,1]$.

$\Rightarrow$ $f'(0) \geq 0$ $\to$ First Minkowski:

$f''(0) \leq 0$, $\to$ Second Minkowski:

(i) $V(A, B, \dots, B)^d \geq V(A) \cdot V(B)^{d-1}$

$$f(\lambda) = \mathrm{vol}_d(\lambda A + (1-\lambda)B)^{1/d} - \lambda \cdot \mathrm{vol}(A)^{1/d} - (1-\lambda)\cdot \mathrm{vol}_B^{1/d}.$$

Let $g(\lambda) := \mathrm{vol}_d(\lambda A + (1-\lambda)B)$.

$$g(\lambda) = \sum_{k=0}^{d} \binom{d}{k} \cdot \lambda^k \cdot (1-\lambda)^{d-k} \cdot V(\underbrace{A, \dots, A}_{k}, \underbrace{B, \dots, B}_{d-k})$$

$$f'(0) = \frac{1}{d} \cdot g(0)^{\frac{1}{d}-1} \cdot g'(0) - \mathrm{vol}(A)^{1/d} + \mathrm{vol}(B)^{1/d}.$$

$$= \frac{1}{d} V(B)^{\frac{1}{d}-1} \cdot \left[\cancel{-d \cdot V(B)} + d \cdot V(A, \dots B)\right]$$

$$- \mathrm{vol}(A)^{1/d} + \cancel{\mathrm{vol}(B)^{1/d}}.$$

$$= V(B)^{1/d - 1} \cdot V(A, B, \dots, B) - V(A)^{1/d}. \quad \underline{\geq 0}$$

Rearrange $\rightsquigarrow V(A, B, \dots, B)^d \geq V(A) \cdot V(B)^{d-1}$.

(ii) $V(A, B, \ldots, B)^d \geq V(A) \cdot V(B)^{d-1}$

(ii) $V(A,B,B,\dots,B)^2 \geq V(A,A,B,\dots,B)\cdot V(B,B,\dots,B)$

$$f(\lambda) = \mathrm{vol}_d(\lambda A + (1-\lambda)B)^{1/d} - \lambda\cdot \mathrm{vol}(A)^{1/d} - (1-\lambda)\cdot \mathrm{vol}_B^{1/d}.$$

Let $g(\lambda) := \mathrm{vol}_d(\lambda A + (1-\lambda)B)$.

$$f(\lambda) = g(\lambda)^{1/d} - (\text{linear terms})$$

$$f''(0) = \frac{d}{d\lambda}\left[\frac{1}{d}g(\lambda)^{\frac{1}{d}-1}\cdot g'(\lambda)\right]\Big|_{\lambda=0}$$

$$= \frac{1}{d}g(\lambda)^{\frac{1}{d}-1}\cdot g''(\lambda) + \frac{1}{d}\left(\frac{1}{d}-1\right)g(\lambda)^{\frac{1}{d}-2}\cdot\left[g'(\lambda)\right]^2\Big|_{\lambda=0}$$

≤ 0

$$\boxed{g(0)\cdot g''(0) - \left(1-\frac{1}{d}\right)\cdot g'(0)^2 \leq 0}$$

(ii) $V(A,B,B,\dots,B)^2 \geq V(A,A,B,\dots,B)\cdot V(B,B,\dots,B)$

$$\boxed{g(0)\cdot g''(0) - \left(1-\frac{1}{d}\right)\cdot g'(0)^2 \leq 0}$$

$$g(\lambda) = \sum_{k=0}^{d} \binom{d}{k} \lambda^k \cdot (1-\lambda)^{d-k}\, V(\underbrace{A,\dots,A}_{k},\underbrace{B,\dots,B}_{d-k})$$

$$g(0) = V(B)$$

$$g'(0) = -d\cdot V(B) + d\cdot V(A,B,\dots,B)$$

$$g''(0) = d\cdot(d-1)\cdot V(B) - \binom{d}{1}\cdot(d-1)\cdot 1\cdot V(A,B,\dots B)$$
$$+\binom{d}{2}\cdot 2\cdot V(A,A,B,\dots,B).$$

$$g(0) = V(B)$$

$$g'(0) = -d \cdot V(B) + d \cdot V(A, B, \dots, B)$$

$$g''(0) = d \cdot (d-1) \cdot V(B) - 2\binom{d}{1}(d-1) \cdot 1 \cdot V(A, B, \dots, B) + \binom{d}{2} \cdot 2 \cdot V(A, A, B, \dots, B).$$

$$g(0) \cdot g''(0) = \cancel{d \cdot (d-1)} \cdot \underline{V(B)^2} - \underline{2\, \cancel{d(d-1)} \cdot V(A, B, \dots, B) \cdot V(B)} + \cancel{d \cdot (d-1)} \cdot V(A, A, B, \dots, B) \cdot V(B)$$

$$\left(1 - \frac{1}{d}\right) \cdot \left[g'(0)\right]^2 = \cancel{d \cdot (d-1)} \cdot \left[\underline{V(B)^2} + \underline{2\, V(B)\, V(A, \dots, B)} + V(A, \dots, B)^2\right]$$

$$\Longrightarrow \quad V(A, A, B, \dots, B) \cdot V(B, B, \dots, B) \leq V(A, B, \dots, B)^2.$$

Recap

Brunn-Minkowski's Theorem:

$$\mathrm{vol}_d(\lambda A+(1-\lambda)B)^{1/d} \geq \lambda\cdot \mathrm{vol}_d(A)^{1/d} + (1-\lambda)\cdot \mathrm{vol}_d(B)^{1/d}.$$

Mixed volumes:

$$\mathrm{vol}_d(\lambda_1 P_1+\dots+\lambda_n P_n) = \sum_{j_1,\dots,j_d} (\lambda_{j_1}\cdots\lambda_{j_d})\cdot V(P_{j_1},\dots,P_{j_d})$$

First and second Minkowski inequalities:

$$V(A,B,\dots,B)^d \geq V(A,A,\dots,A)\cdot V(B,B,\dots,B)^{d-1}$$

$$V(A,B,\dots,B)^2 \geq V(A,A,B,\dots,B)\cdot V(B,B,\dots,B).$$

THE END