Reading Group (W21)
Sector Stable Polynomials

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Agenda

- Introduction
- Background on HDX
- Spectral independence
- Sector-stable polynomials
- Application to sampling planar matchings
- Bonus

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Motivation

- Given a distribution $\mu: 2^{[n]} \to \mathbb{R}_{\geq 0}$
- Task: efficiently sample set $S \subseteq [n]$ according to μ

- Question: what can properties of the *generating polynomial*

$$g_\mu(z) := \sum_S \mu(S) z^S$$

  inform us about sampling?

A fuzzy roadmap

$\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$

polynomial

$g_\mu(z) = \sum \mu(S) z^S$

Unknown property 1

Sector stability

HDX

Completely log-concave

"local" walks

mix VERY well

"down-up walk"

mix well

Unknown property 2

spectral independence

(="fractional log-concavity")

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Background on HDX

- Simplicial complex:

$X \subseteq 2^{[n]}$

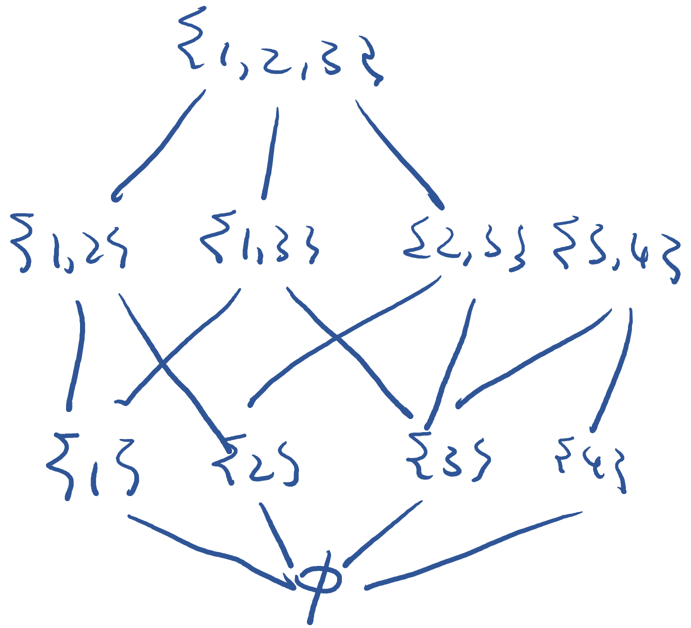
downward-closed.

dimension of X : (# of elements of biggest face $\sigma \in X$) -1.

dim 0

dim 1

dim 2



- Pure simplicial complex:

all maximal faces have same size

Background on HDX

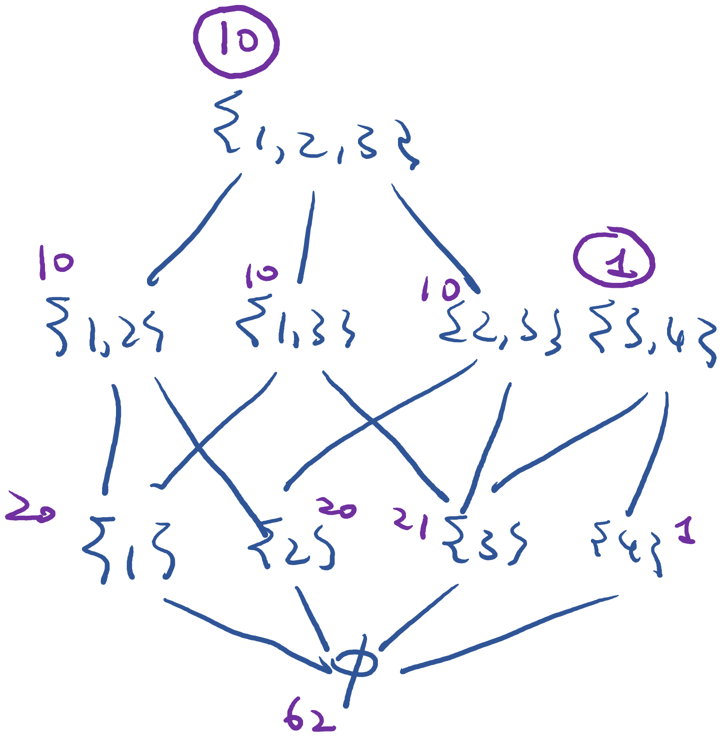
- Weights on faces:

$$\omega : X \to \mathbb{R}_{\geq 0}$$

balanced:

for any $\sigma \in X$,

$$\omega(\sigma) = \sum_{\substack{\tau \supseteq \sigma \\ |\tau| = |\sigma|+1}} \omega(\tau)$$



Background on HDX

- Links and walks on links:

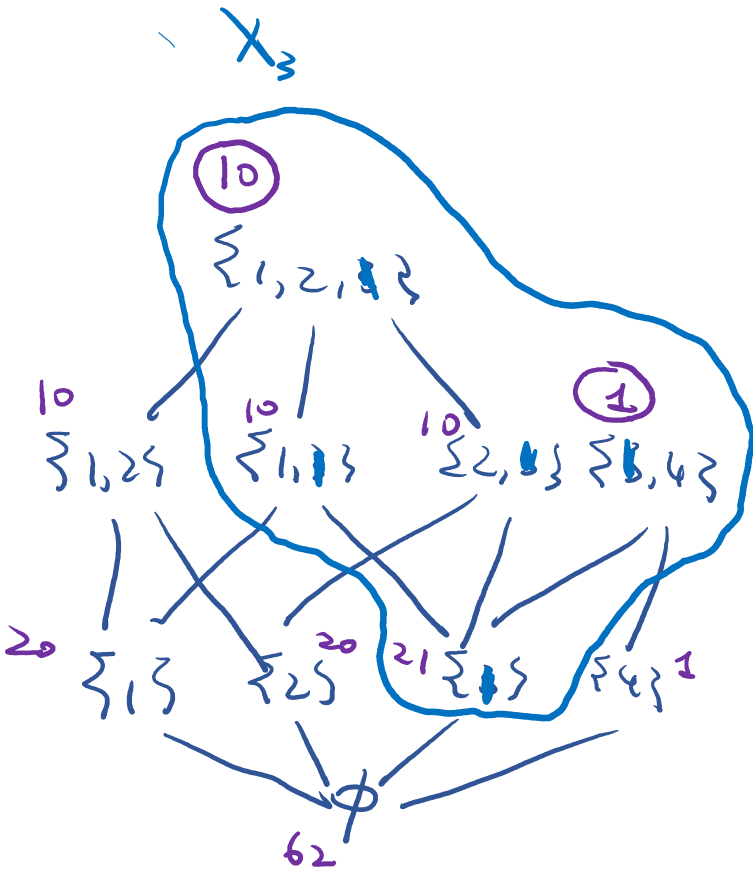
X : dim d

$\sigma \in X : |\sigma| \leq d-1$

$X_\sigma := \{\tau \setminus \sigma : \tau \supseteq \sigma,\ \tau \in X\}$

$X_\phi = X$,

Walks on links: take 1-faces as vertices
2-faces as edges
weight given by w



Background on HDX

- α-expander:

  for any $\sigma \in X$, $|\sigma| \leq d-1$,
  walk on link X_σ has $\lambda_2 \leq \alpha$.

- $(\alpha_0, \alpha_1, \dots, \alpha_{d-2})$-expander:

  for any $\sigma \in X$, $|\sigma| = k+1$,
  walk on link X_σ has $\lambda_2 \leq \alpha_k$.

Down-up walk

$k > l$

- $k \leftrightarrow l$ down-up walk:
  - Start with $S_0 \in X$ of size k
  - Repeat:
    - Choose uniformly random $T_{i+1} \subset S_i$ of size l (down)
    - Choose $S_{i+1} \supset T_{i+1}$ of size k, according to $w(S)$ (up)

- [~~Opp~~ KO 18] Down-up walks on HDX mix quickly
- This is the sampling algorithm that we'll analyze in this talk

Examples

- Matroids

maximal faces of X_M = bases of M

X_M is 0-expander.

- Expander graphs

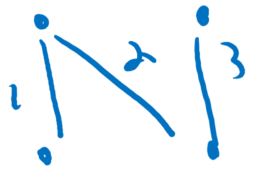
nodes of $G \longrightarrow$ 1-face of X_G

edges of $G \longrightarrow$ 2-face of X_G

- Matchings

ϕ $\{1\}$ $\{2\}$ $\{3\}$

$\{1,3\}$



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Correlation/Influence matrix

$\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$

$$\mathcal{I}_\mu^{\text{inf}}(i,j) := \begin{cases} 0, & i=j \\ P[j|i] - P[j|\bar{i}], & i \neq j \end{cases}$$

$$\mathcal{I}_\mu^{\text{cor}}(i,j) := \begin{cases} 1-P[i], & i=j \\ P[j|i] - P[j], & i \neq j. \end{cases}$$

Spectral independence

α spectrally independent.

$$\lambda_{\max}\left(\Psi_\mu^{cor}\right) \leq \alpha$$

Consider row sum instead.

$$\sum_j \left|\Psi_\mu^{cor}(i,j)\right| \leq \alpha.$$

Why spectral independence is good

[AL20], [ALO20]

Spectral independence (μ and its conditionals)

⇒ bound on λ_2 of down-up walk

⇒ fast mixing!

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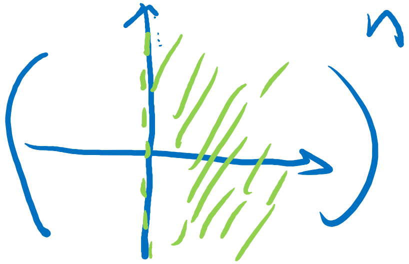
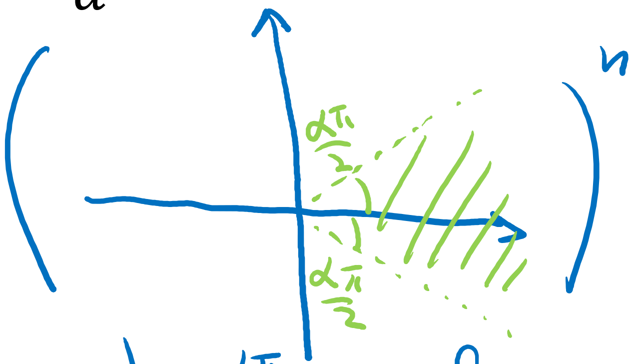
What is a sector-stable polynomial?

- Stable: no roots in upper half space
- Hurwitz stable: no roots in right half space

[$f(z_1,\dots,z_n)$, $\mathrm{Re}(z_1),\dots,\mathrm{Re}(z_n) > 0 \Rightarrow f(z_1,\dots,z_n) \neq 0$]



- Sector stable: no roots in Γ_α^n



[$f(z_1,\dots,z_n)$, $|\mathrm{Arg}(z_i)| < \frac{\alpha\pi}{2} \Rightarrow f(z_1,\dots,z_n) \neq 0$.]

* Hurwitz stability $= \Gamma_1$-stability.

Why do we care?

- Because of this main theorem:

Theorem 50. *Consider a multi-affine $f \in \mathbb{R}_{\geq 0}[z_1, \cdots, z_n]$ polynomial that is Γ_α-stable with $\alpha \leq 1$. Let $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be the distribution generated by f, then Ψ_μ^{inf} and Ψ_μ^{cor} have bounded row norms. Specifically,*

$$\sum_j |\Psi_\mu^{inf}(i,j)| \leq 2/\alpha - 1,$$

and

$$\sum_j |\Psi_\mu^{cor}(i,j)| \leq 2/\alpha.$$

As a corollary, the same bounds hold for maximum eigenvalues, i.e., $\lambda_{\max}(\Psi_\mu^{inf}) \leq 2/\alpha - 1$ and $\lambda_{\max}(\Psi_\mu^{cor}) \leq 2/\alpha$.

- "Sector stability implies spectral independence"

Proof

$$\Phi_\mu^{inf}(i,j) = \begin{cases} 0 , & i=j \\ P[j|i] - P[j|\bar{i}] , & i \neq j \end{cases}$$

- Step 1: rewrite the row sum as $\phi'(0)$

(say $i=n$) $f = g + w_n h$, $g = f|_{w_n=0}$, $h = \partial_n f$.

$$\sum_{j\neq n} \left| \Phi_\mu^{inf}(n,j) \right| = \sum_{j \neq n} \left| P[j|n] - P[j|\bar{n}] \right|$$

$$= \sum_{j\neq n} \left| \frac{\partial_j h(1)}{h(1)} - \frac{\partial_j g(1)}{g(1)} \right| \quad \text{—} (*)$$

$$P[j|n] = \frac{P[j \wedge n]}{P[n]} = \frac{\partial_j h(1)}{h(1)}$$

Consider this transformation

$$w_j = \begin{cases} y , & \text{if } P[j|n] - P[j|\bar{n}] < 0 \\ y^{-1} , & \text{if } P[j|n] - P[j|\bar{n}] \geq 0 . \end{cases}$$

$$\boxed{\phi(z) = \log\left(\frac{\bar{g}(e^z)}{\bar{h}(e^z)}\right) - \log\left(\frac{\bar{g}(1)}{\bar{h}(1)}\right)}$$

Define $\bar{g}(y) = g(w)$, $\bar{h}(y) = h(w)$

Verify that $(*) = \frac{\bar{g}'(1)}{\bar{g}(1)} - \frac{\bar{h}'(1)}{\bar{h}(1)} = \log \bar{g}(1)' - \log \bar{h}(1)' = \phi'(0)$

Proof

- Step 2: recall Schwarz's lemma

$$\phi : D \longrightarrow D \quad (D := \{|z| < 1\})$$

s.t. ϕ is holomorphic, $\phi(0) = 0$.

Then $|\phi(z)| \leq |z|$.

$\Rightarrow |\phi'(0)| \leq 1$.

Want to modify our function so that we can apply Schwarz's lemma

Proof

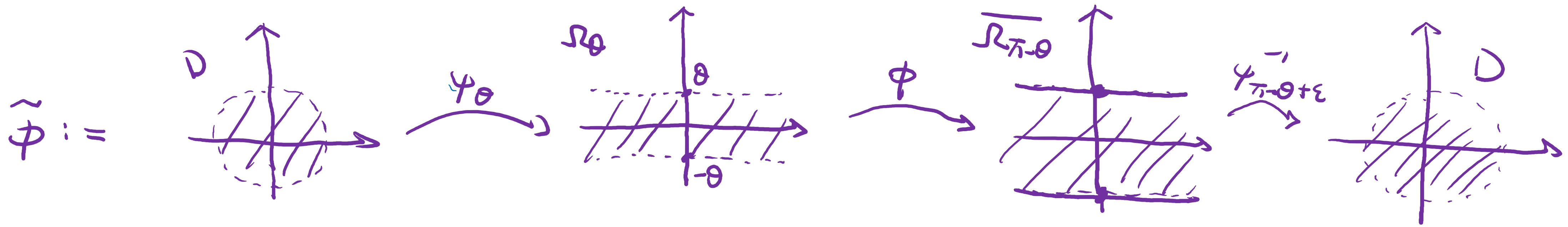
$\bar{g}(y) = g(u)$

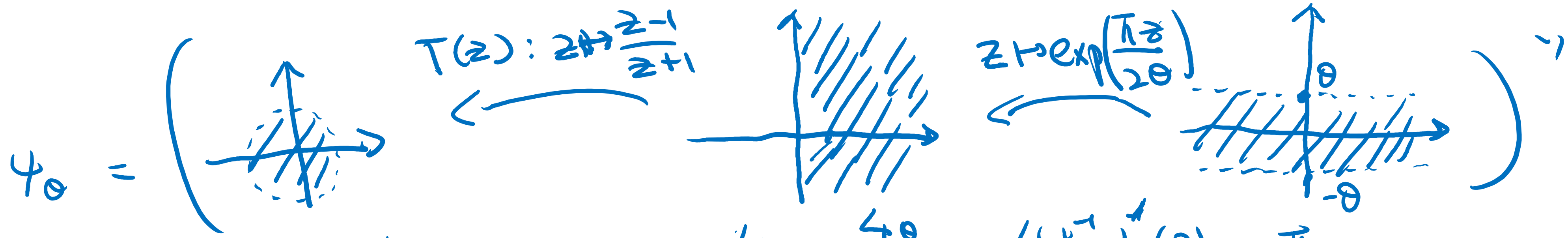
$$w_j = \begin{cases} y, & P[j|n] - P[j|\bar{n}] < 0 \\ y^{-1}, & P[j|n] - P[j|\bar{n}] \geq 0 \end{cases}$$

$$\phi(z) = \log\left(\frac{\bar{g}(e^z)}{\bar{h}(e^z)}\right) - \log\left(\frac{\bar{g}(1)}{\bar{h}(1)}\right)$$

$\phi(0) = 0.$

- Step 3: construct a map $\tilde{\phi}$ from D to D

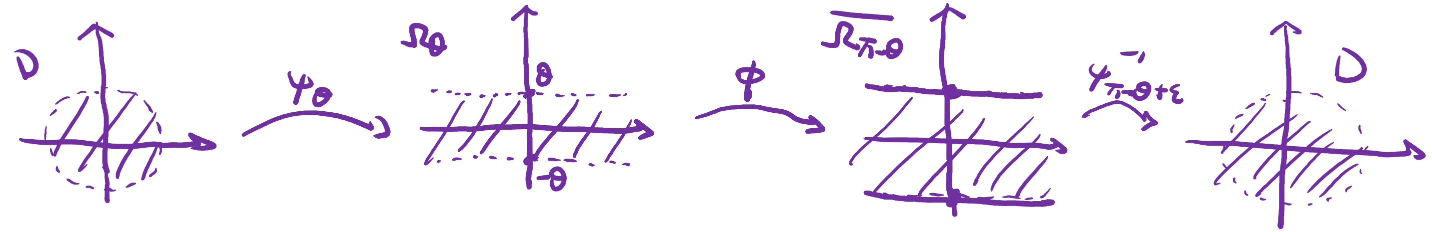




Properties: $\psi_\theta(0) = 0$, $\psi_\theta'(0) = \frac{4\theta}{\pi}$, $(\psi_\theta^{-1})'(0) = \frac{\pi}{4\theta}$.

Proof

$\tilde{\phi} :=$



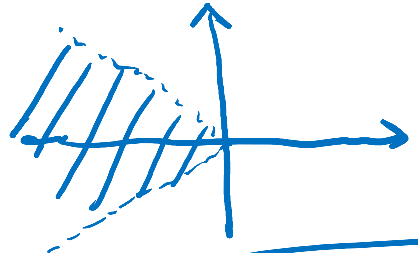
- Step 4: analyze the map $\tilde{\phi}$

$$\phi(y) = \log\left(\frac{\bar{g}(e^z)}{\bar{h}(e^z)}\right) - \log\left(\frac{\bar{g}(1)}{\bar{h}(1)}\right)$$

$z \mapsto e^z$ sends

f Γ_α-stable, take $\theta = \frac{\alpha\pi}{2}$. Then,

$\xi = e^z$ $\quad \frac{\bar{g}(\xi)}{\bar{h}(\xi)}$ doesn't take value in



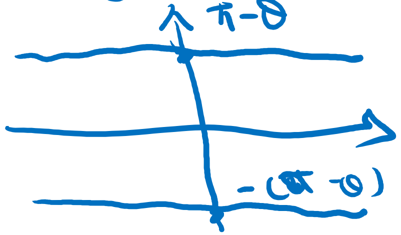
$= -\Gamma_\alpha$

If not, $\exists\, \omega \in -\Gamma_\alpha$ s.t. $\bar{g}(\xi) - \omega \bar{h}(\xi) = 0$.

$\Rightarrow f(\xi^{\pm 1}, \xi^{\pm 1}, \dots, \xi^{\pm 1}, -\omega) = 0$

exponent can be ± 1.

Take log, bring to



Proof

- Step 5: obtain bound on $\phi'(0)$

$$\hat{\phi} = \varphi_{\pi-\theta+\varepsilon}^{-1} \circ \phi \circ \varphi_\theta$$

$$\hat{\phi} : D \to D, \quad \hat{\phi}(0) = 0.$$

Schwarz's lemma $\Rightarrow |\hat{\phi}'(0)| \le 1.$

$$\Rightarrow |\varphi_{\pi-\theta+\varepsilon}^{-1}(0)'| \cdot |\phi'(0)| \cdot |\varphi_\theta'(0)| \le 1$$

$$\frac{\pi}{4(\pi-\theta+\varepsilon)} \cdot |\phi'(0)| \cdot \frac{4\theta}{\pi} \le 1$$

$\theta = \frac{\alpha\pi}{2}$, $\varepsilon \to 0$:

$$|\phi'(0)| \le \frac{\pi-\theta}{\theta}$$

$$= \frac{2}{\alpha} - 1.$$

$\boxtimes$

** The complete framework **

- Given distribution $\mu: 2^{[n]} \to \mathbb{R}_{\geq 0}$
- Algorithm: natural $k \leftrightarrow l$ down-up walk for $k - l = O(1)$
- Analysis:
  - Show that $g_\mu(z) \coloneqq \sum_S \mu(S)\, z^S$ (and conditionals) is Γ_α-stable
  - Sector-stability implies spectral independence
  - Spectral independence implies fast mixing of $k \leftrightarrow l$ down-up walk

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Application: k-planar matchings

- Given a *planar* graph
- Want to sample size-k matchings uniformly at random

- [FKT] Counting perfect matchings in planar graphs is in #P ☺
- [Jer87] Counting size-k matchings is #P-hard ☹

Idea: sample "monomers" first

- Sample vertex set $S \subseteq V$ of size $2k$
- Want:
  - Perfect matching on S
  - No edge incident to vertices in V$\backslash S$
- Corresponding generating polynomial is

$$g(z) \coloneqq \sum_{|S|=2k} m_S z^S$$

- This polynomial (and conditionals) are $\Gamma_{1/2}$-stable!

$\Gamma_{1/2}$-stability of the k-matching polynomial

Here's the strategy -

- Step 1: the (modified) matching polynomial

$$g_0(z) := \sum_{M:matching} z^M = \sum_S m_S z^S$$

  is Hurwitz (Γ_1-) stable

- Step 2: imposing cardinality constraint on Γ_1-stable, constant partiy polynomial makes it $\Gamma_{1/2}$-stable

(+ monomials have the same degree parity.)

Proof

Step 2.

g constant parity, Γ_1-stable

- g_{min}, g_{max} are Γ_1-stable.
- fix $z_1, \ldots, z_n \in \Gamma_{1/2}$ and consider

$$h(z) = g(z_1 z, z_2 z, \ldots, z_n z).$$

Extract degree-k part from this.

Proof

Proof

Proof

Conclusion: a not-so-fuzzy roadmap

WORK

$\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$

Sector stability

[AASV]

Spectral independence

[ALO]

walks of links have good λ_2

[KO], [AL]

$k \leftrightarrow \ell$ walk mixes quickly.

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α- Fractional log-concavity

$f \in \mathbb{R}_{\geq 0}[z_i]$

- Definition: for $z \in \mathbb{R}^n_{>0}$, $\log(f(\{z_i^\alpha\}))$ is concave

- Relation to sector stability:

Lemma 67. *For $\alpha \in [0, 1/2]$, if polynomial $f \in \mathbb{R}_{\geq 0}[z_1, \cdots, z_n]$ is $\Gamma_{2\alpha}$-sector-stable then f is α fractionally log-concave.*

(Proof strategy: relate fractional log-concavity to bound on λ_{max} of correlation matrix)

Sector stability and Newton polytope

Lemma 66. *Let* $\mu : 2^{[n]} \to \mathbb{R}$ *be a* $\Gamma_{1/k}$*-sector-stable distribution, then the length of edges of* $\operatorname{newt}(\mu)$ *is at most* $2k$.

$\operatorname{newt}(\mu) \subseteq \mathbb{R}^n$: $\operatorname{conv}(\{\operatorname{supp}(\mu)\})$

A Conjecture

Remark 77. In Lemma 66, we show the convex hull of the support of a Γ_α-sector stable polynomial has edge length bounded by $O(1/\alpha)$. We can show a similar result for α-fractionally log-concave polynomial. We leave the problem of characterizing the support of α-fractionally log-concave polynomial to future work.

Conjecture 22. *Suppose that μ is the uniform distribution on a subset of the hypercube $F \subseteq \{0,1\}^n$, such that the convex hull $\operatorname{conv}(F)$ has edges of bounded length $O(1)$. Then we conjecture that the polynomial*

$$\sum_{S \in F} \mu(S) \prod_{i \in S} z_i$$

is fractionally log-concave for a parameter $\alpha > \Omega(1)$.

Related questions

- Characterize the support of sector-stable/fractionally log-concave polynomials

- Γ_α-stable polynomials are $\frac{\alpha}{2}$-fractionally log-concave. Is there a reverse implication?

End