Reading Group W21
Lorentzian Polynomials (Part III)

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Agenda

- Proof of Mason’s conjecture
- Preservers of Lorentzian polynomials

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- Proof of Mason’s conjecture
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Mason's conjectures

- These are three statements of increasing strength:

Conjecture 4.13. For any matroid M on $[n]$ and any positive integer k,

(1) $I_k(\mathrm{M})^2 \geqslant I_{k-1}(\mathrm{M}) I_{k+1}(\mathrm{M})$,

(2) $I_k(\mathrm{M})^2 \geqslant \frac{k+1}{k} I_{k-1}(\mathrm{M}) I_{k+1}(\mathrm{M})$,

(3) $I_k(\mathrm{M})^2 \geqslant \frac{k+1}{k} \frac{n-k+1}{n-k} I_{k-1}(\mathrm{M}) I_{k+1}(\mathrm{M})$,

where $I_k(\mathrm{M})$ is the number of k-element independent sets of M.

Tutte Polynomial

- Given M: matroid on $[n]$, $rk_M : 2^{[n]} \to \mathbb{N}$ its rank function
- Consider the polynomial in H_n^k

$$Z_{q,M}^k(w) = \sum_{A \in \binom{n}{k}} q^{-rk_M(A)} w^A$$

- Define the *homogeneous multivariate Tutte polynomial of M* by

$$Z_{q,M}(w_0, w) = \sum_{k=0}^{n} Z_{q,M}^k(w)\, w_0^{n-k}$$

How it relates to ULC of $I_k(M)$

- $Z^k_{q,M}(w) = \sum_{A\in\binom{n}{k}} q^{-rk_M(A)} w^A$
- $Z_{q,M}(w_0, w) = \sum_{k=0}^n Z^k_{q,M}(w)\, w_0^{n-k}$

- Want to “kill” dependent sets, those with $rk_M(A) < |A|$

- Diagonalize and use property of *bivariate* Lorentzian polynomials

Main Theorem

Theorem 4.10. For any matroid M and $0 < q \leqslant 1$, the polynomial $\mathrm{Z}_{q,\mathrm{M}}$ is Lorentzian.

Lemma 4.11. The support of $Z_{q,\mathrm{M}}$ is M-convex for all $0 < q \leqslant 1$.

Whiteboard

$$Z^k_{q,M}(w) = \sum_{A \in \binom{[n]}{k}} q^{-r_M(A)} w^A$$

$$Z_{q,M}(w) = \sum_{k=0}^{n} Z^k_{q,M}(w) \cdot w_0^{n-k}$$

$$\{(\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1} : \alpha_0 + \ldots + \alpha_n = n\}$$

$$\cap \; \mathbb{N} \times [0,1] \times [0,1] \times \ldots \times [0,1]$$

- $Z^k_{q,M}(w) = \sum_{A\in\binom{n}{k}} q^{-rk_M(A)} w^A$
- $Z_{q,M}(w_0, w) = \sum_{k=0}^n Z^k_{q,M}(w)\, w_0^{n-k}$

Whiteboard

- Strategy: Induction on n.

- For $i \in [n]$, $\partial_i Z_{q,M} = Z_{q,M/i} \cdot q^{-rk_M(i)}$

  $(rk_M(A) = rk_{M/i}(A \setminus i) + rk_M(i))$

- $\partial_0^{n-2} Z_{q,M} = (n-2)!\left[\frac{n(n-1)}{2} w_0^2 + (n-1) Z^1_{q,M} w_0 + Z^2_{q,M}\right]$

- $\Delta \geq 0 \iff (Z^1_{q,M}(w))^2 \geq 2\frac{n}{n-1}\left(Z^2_{q,M}(w)\right)$

Whiteboard

- $Z^k_{q,M}(\color{red}{w}) = \sum_{A\in\binom{n}{k}} q^{-rk_M(A)}\color{red}{w^A}$
- $Z_{q,M}(w_0,\color{red}{w}) = \sum_{k=0}^{n} Z^k_{q,M}(\color{red}{w})\, w_0^{n-k}$

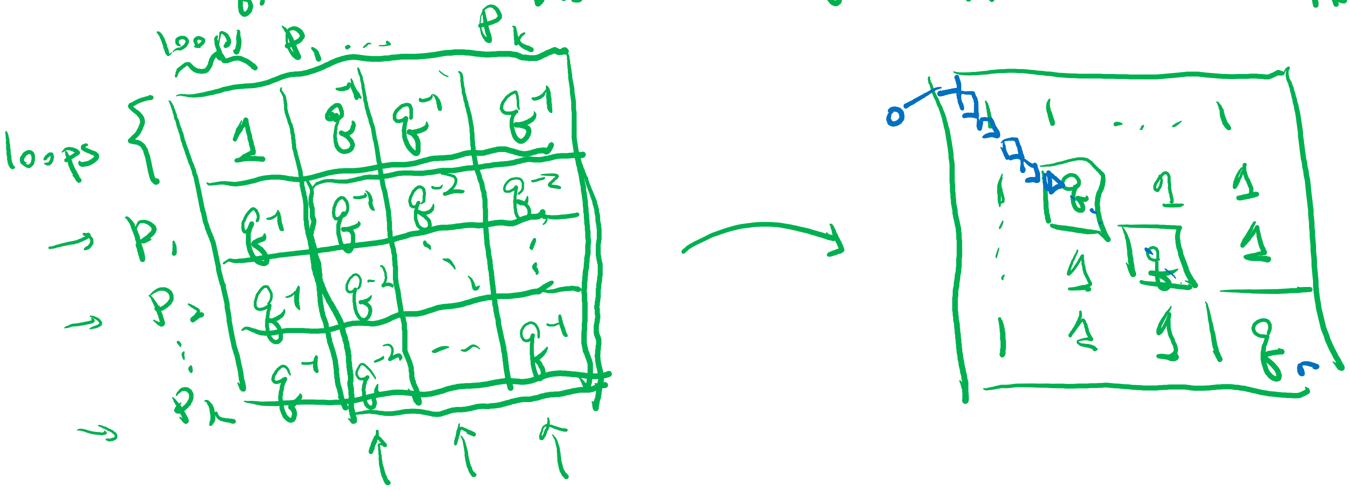
- Mapping: $w_i \mapsto q w_i$ if $rk_M(i) = 1$

Then:

$$Z^1_{q,M}(w) = w_1 + w_2 + \dots + w_n =: e^1_{[n]}(w)$$

$$Z^2_{q,M}(w) = e^2_{[n]}(w) - (1-q)\left(e^2_{P_1}(w) + \dots + e^2_{P_k}(w)\right)$$

P_i: parallel classes.



Whiteboard

- $Z_{q,M}^{k}(w) = \sum_{A\in\binom{n}{k}} q^{-rk_M(A)} w^A$
- $Z_{q,M}(w_0, w) = \sum_{k=0}^{n} Z_{q,M}^{k}(w)\, w_0^{n-k}$

- Need to check $\left(Z_{q,M}^{1}(w)\right)^2 \geq 2\frac{n}{n-1} Z_{q,M}^{2}(w)$, or $\left(e_{[n]}^{1}(w)\right)^2 \geq 2\frac{n}{n-1}\left[e_{[n]}^{2}(w) - (1-q)\sum_i e_{P_i}^{2}(w)\right]$

- Case 1: $\sum_i e_{P_i}^{2}(w) \geq 0$. Check $n\sum w_i^2 \geq \left(\sum w_i\right)^2$.

- Case 2: $\sum_i e_{P_i}^{2}(w) < 0$. Check $n\left(\sum_{rk(i)=0} w_i^2 + \sum_i \left(e_{P_i}^{1}(w)\right)^2\right) \geq \left(e_{[n]}^{1}(w)\right)^2$.

$w_1^2 + \dots + w_j^2 + (w_{j+1} + \dots + w_{j'})^2 + \dots + (w \dots)^2$

If there are N parts,

$N(\quad) \geq (w_1 + \dots + w_n)^2$

$\Rightarrow n(\dots) \geq (w_1 + \dots + w_n)^2$

- $Z_{q,M}^k(w) = \sum_{A \in \binom{n}{k}} q^{-rk_M(A)} w^A$
- $Z_{q,M}(w_0, w) = \sum_{k=0}^n Z_{q,M}^k(w)\, w_0^{n-k}$

Wrapping up the proof

- $A \subseteq [n]$ is dependent iff $rk_M(A) < |A|$

$q^{-rk_M(A)} \cdot q^{|A|} \cdot w^A$

- Key is to consider the polynomial $f_M(w_0, w) = \lim_{q \to 0+} Z_{q,M}(w_0, qw)$

$= \sum_{k=0}^{n} \left(\sum_{A \in I_k} w^A \right) w_0^{n-k}$

- Setting $w_1 = w_2 = \cdots = w_n = z$, we end up with a bivariate Lorentzian polynomial $g_M(w_0, z) = \sum_{k=0}^n I_k(M)\, z^k w_0^{n-k}$

log-concave !!!

- Coefficient sequence is ULC, so done :)

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- Proof of Mason's conjecture
- Preservers of Lorentzian polynomials

Motivation

- Showing that certain polynomials are Lorentzian
- Creating new Lorentzian polynomials from existing ones

What we already know...

- $f(w)$ is Lorentzian $\Rightarrow$ $f(Aw)$ is Lorentzian, $(A \geq 0)$
- f, g is Lorentzian $\Rightarrow$ fg is Lorentzian
- Bunch of stability preserving operators.

Homogeneous operator and its symbol

- Let $T: \mathbb{R}_{\kappa}[w_i] \to \mathbb{R}_{\gamma}[w_i]$ be a linear operator

  consider polynomial whose monomials w^{β} satisfy $0 \leq \beta_i \leq \kappa_i$

- It is *homogeneous of degree l* if it maps any monomial of degree d to zero or to a monomial of degree $d + l$

- The symbol of T is defined as

$$sym_T(w, u) := \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} T(w^{\alpha}) u^{\kappa - \alpha}$$

Main theorems of the day

$$\mathrm{sym}_T = \sum_{0 \le \alpha \le \kappa} \binom{\kappa}{\alpha} T(w^\alpha) \cdot u^{\kappa-\alpha}$$

$k = |\kappa|_1$

Theorem 3.2. If $\mathrm{sym}_T \in \mathrm{L}_{m+n}^{k+\ell}$ and $f \in \mathrm{L}_n^d \cap \mathbb{R}_\kappa[w_i]$, then $T(f) \in \mathrm{L}_m^{d+\ell}$.

Theorem 3.4. If T is a homogeneous linear operator that preserves stable polynomials and polynomials with nonnegative coefficients, then T preserves Lorentzian polynomials.

More stable polynomial facts

- Given stable $f, g \in \mathbb{R}[w_1, \dots, w_n]$
- Write $f \prec g$ if $g + w_{n+1} f$ is stable

"in proper position"

Lemma 2.9. Let f, g_1, g_2, h_1, h_2 be stable polynomials satisfying $h_1 < f < g_1$ and $h_2 < f < g_2$.

(1) The derivative $\partial_1 f$ is stable and $\partial_1 f < f$.

(2) The diagonalization $f(w_1, w_1, w_3, \dots, w_n)$ is stable.

(3) The dilation $f(a_1 w_1, \dots, a_n w_n)$ is stable for any $a \in \mathbb{R}^n_{\geqslant 0}$.

(4) If f is not identically zero, then $f < \theta_1 g_1 + \theta_2 g_2$ for any $\theta_1, \theta_2 \geqslant 0$.

(5) If f is not identically zero, then $\theta_1 h_1 + \theta_2 h_2 < f$ for any $\theta_1, \theta_2 \geqslant 0$.

Proof of Theorem 3.2

- Prove a special case first:

Lemma 3.3. Let $T = T_{w_1,w_2} : \mathbb{R}_{\overset{n}{(1,\ldots,1)}}[w_i] \to \mathbb{R}_{\overset{m}{(1,\ldots,1)}}[w_i]$ be the linear operator defined by

$$T(w^S) = \begin{cases} w^{S\setminus 1} & \text{if } 1 \in S \text{ and } 2 \in S, \\ w^{S\setminus 1} & \text{if } 1 \in S \text{ and } 2 \notin S, \\ w^{S\setminus 2} & \text{if } 1 \notin S \text{ and } 2 \in S, \\ 0 & \text{if } 1 \notin S \text{ and } 2 \notin S, \end{cases} \quad \text{for all } S \subseteq [n].$$

Then T preserves the Lorentzian property.

$$\mathrm{sym}_T(w,u) = \sum_{A \subseteq [n]} T(w^A) \cdot u^{[n]\setminus A} = \sum_{A' \subseteq \{3,\ldots,n\}} w^{A'} \left(w^{\{2\}} \cdot u^{[n]-A'-\{1,2\}} + 1 \cdot u^{[n]-A'-\{1\}} + 1 \cdot u^{[n]-A'-\{2\}} \right)$$

$$\left(A = (\phi / \{1\}) \cup (\phi / \{2\}) \cup A'\right)$$

$0 \le \alpha \le \kappa$

$$= \left[\sum_{A' \subseteq \{3,\ldots,n\}} w^{A'} u^{\{3,\ldots,n\}-A'}\right] (w_2 + u_2 + u_1)$$

Theorem 3.2. If $\mathrm{sym}_T \in \mathrm{L}^{k+\ell}_{m+n}$ and $f \in \mathrm{L}^d_n \cap \mathbb{R}_\kappa[w_i]$, then $T(f) \in \mathrm{L}^{d+\ell}_m$.

Whiteboard

Proof of Lemma 3.3:

Let $f \in \mathbb{R}_{(1,\dots,1)}[w_1,\dots,w_n] \cap \mathrm{L}^{\circ\, d+1}_n$.

- Check $\mathrm{supp}(T(f))$ is M-convex.

$$\mathrm{supp}(T(f)) = (\{0,1\} \times \{0,1\} \times \dots \times \{0,1\}) \cap \{\alpha : |\alpha|_1 = d\}$$

- Check $\partial^S(T(f))$ is Lorentzian / stable for $|S| = d-2$.

If $1 \in S$, $\partial^S(T(f)) = 0$

If $1 \notin S$ ... $f = h + w_1 \partial_1 f$. Then $T(f) = \partial_2 h + \partial_1 f$

~~If $2 \in S$ ... $\partial_2 f = \partial_2 h + w_1 \partial_1 \partial_2 f$~~

~~$2 \notin S$~~, ...

If $2 \in S$, $\partial_2 T(f) = \partial_1 \partial_2 f$ which means $\partial^S T(f) = \partial^{S \cup 1} f$.

If $2 \notin S$, $\partial^S T(f) = \partial^S \partial_2 h + \partial^S \partial_1 f$.

$\partial^S \partial_1 \partial_2 f \prec \partial^S \partial_1 f$ and $\partial^S \partial_1 \partial_2 f \prec \partial^S \partial_2 h$

Theorem 3.2. If $\mathrm{sym}_T \in \mathrm{L}^{k+\ell}_{m+n}$ and $f \in \mathrm{L}^d_n \cap \mathbb{R}_\kappa[w_i]$, then $T(f) \in \mathrm{L}^{d+\ell}_m$.

Whiteboard

Polarization: $\Pi^\uparrow_\kappa : \mathbb{R}_\kappa[w_i] \to \mathbb{R}^a_\kappa[w_{ij}]$ in $k = |\kappa|$ variables.

$$w_1^{\alpha_1} \cdots w_n^{\alpha_n} \mapsto \binom{\kappa}{\alpha}^{-1} \cdot \prod_{i=1}^{n} e^{\alpha_i}_{[w_{ij}]}$$

$w_{i1}, \dots, w_{i\kappa_i}$.

Projection: $\Pi^\downarrow_\kappa : \mathbb{R}^a_\kappa[w_{ij}] \to \mathbb{R}_\kappa[w_i]$

$$w_{ij} \mapsto w_i$$

Polarization of operator T:

$$\begin{tikzcd} \mathbb{R}_\kappa[w_i] \arrow[r] & \mathbb{R}_\gamma[w_i] \arrow[d] \\ \mathbb{R}^a_\kappa[w_{ij}] \arrow[u] \arrow[r] & \mathbb{R}^a_\gamma[w_{ij}] \end{tikzcd}$$

$$\Pi^\uparrow T := \Pi^\uparrow_\gamma \circ T \circ \Pi^\downarrow_\kappa$$

Key property: "polarization of symbol = symbol of polarized poly"

$$\Pi^\uparrow_{\kappa \oplus \gamma} \mathrm{sym}_T = \mathrm{sym}_{\Pi^\uparrow T}.$$

Theorem 3.2. If $\mathrm{sym}_T \in \mathrm{L}_{m+n}^{k+\ell}$ and $f \in \mathrm{L}_n^d \cap \mathbb{R}_\kappa[w_i]$, then $T(f) \in \mathrm{L}_m^{d+\ell}$.

Whiteboard

Consider $f(v) \in L_n^d$ multiaffine.

$\mathrm{sym}_T(w,u) \in L_{m+n}^{\ell+n}$. Target: show that $T(f) \in L_n^{d+\ell}$.

- $\mathrm{sym}_T(w,u) = \sum_{0\le\alpha\le\kappa} T(w^\alpha)\cdot u^{\kappa-\alpha}$
- $\sum_{0\le\alpha\le\kappa} T(w^\alpha)\cdot \underline{u^{\kappa-\alpha}} \cdot \underline{f(v)}$ is Lorentzian.
- Let's denote operator by T_{u_i,v_i},

$$\left(\prod_{i=1}^{n} T_{u_i,v_i}\right)\left(\sum_{0\le\alpha\le\kappa} T(w^\alpha)\cdot u^{\kappa-\alpha}\cdot f(v)\right)$$

$$f = \sum c_\alpha w^\alpha$$

$$= \sum_{0\le\alpha\le\kappa} T(w^\alpha)\cdot \partial^\alpha f(v).$$

$$T(f) = \sum c_\alpha T(w^\alpha)$$

- Set $v_i = 0 \Rightarrow$ get $\sum c_\alpha T(w^\alpha) = T(f)$.

Theorem 3.4. If T is a homogeneous linear operator that preserves stable polynomials and polynomials with nonnegative coefficients, then T preserves Lorentzian polynomials.

Proof of Theorem 3.4

- Let's invoke a blackbox:

Proof. According to [BB09, Theorem 2.2], T preserves stable polynomials if and only if either

(I) the rank of T is not greater than two and T is of the form

$$T(f) = \alpha(f)P + \beta(f)Q,$$

where α, β are linear functionals and P, Q are stable polynomials satisfying $P \prec Q$,

(II) the polynomial $\mathrm{sym}_T(w, u)$ is stable, or

(III) the polynomial $\mathrm{sym}_T(w, -u)$ is stable.

Consequence #1

- If $f \in L_n^d$, then $MAP(f) \in L_n^d$

Consequence #2

- The *normalization* of $f \in L_n^d$ (denoted by $N(f)$) is again Lorentzian

$f = \sum c_\alpha w^\alpha$, $N(f) = \sum \frac{c_\alpha}{\alpha!} w^\alpha$.

Consequence #3

- If $N(f)$ and $N(g)$ are Lorentzian then so is $N(fg)$

$N(h) \longmapsto N(gh)$

Wrapping it all up...