Reading Group W21
Lorentzian Polynomials (Part II)

Alex Tung

17 February, 2021

Recall

Last time, we have:

- Defined Lorentzian polynomial, and discussed different ways to define/understand the polynomial class
- Discussed its relations with other polynomial classes
- Briefly talked about the discrete side of things – via M-convexity
- Introduced a key property (Hodge-Riemann relation) of H_f

Today we will talk about more advanced topics :)

Bug fix

$$\beta \qquad\qquad\qquad\qquad \alpha$$

$$\{ \underline{(2,0,0)}, (1,1,0), \underline{(0,1,1)} \}$$

$$\{ (2,0,0), (1,0,1), (0,1,1) \}$$

M-convex: for any $i \in [n]$ s.t.

$\alpha_i > \beta_i$

$\exists\ j \in [n]$ s.t.

$\alpha_j < \beta_j$, and

$\alpha_j - e_i + e_j \in J$

$i = 2$

$$(0,1,1) + e_j - (0,1,0)$$

forced to choose $j=1$ s.t. $\alpha_j < \beta_j$

$$(0,1,1) + (1,0,0) - (0,1,0) = (1,0,1)$$

Today's menu

- c-Rayleigh property
- Generating polynomial of M-convex sets
- CLC $\Leftrightarrow$ Lorentzian
- Proof of Mason's conjecture

Agenda

- c-Rayleigh property
- Generating polynomial of M-convex sets
- CLC $\Leftrightarrow$ Lorentzian
- Proof of Mason's conjecture

What is c-Rayleigh?

- Let $f \in \mathbb{R}[w_1, \dots, w_n]$
  - coefficients nonnegative
  - not necessarily homogeneous
- Given $c > 0$, f is c-Rayleigh if

$$\partial^\alpha f(w) \cdot \partial^{\alpha+e_i+e_j} f(w) \le c \cdot \partial^{\alpha+e_i} f(w) \cdot \partial^{\alpha+e_j} f(w)$$

for any $\alpha \in \mathbb{N}^n, i, j \in [n], w \in \mathbb{R}^n_{\ge 0}$

$\Longleftrightarrow$ $P[i \in S, j \in S] \le c \cdot P[i \in S] \cdot P[j \in S]$

$g(w) \cdot \partial_i \partial_j g(w) \le c \cdot \partial_i g(w) \cdot \partial_j g(w)$

- In other words, $\partial^\alpha f$ has some sort of negative dependence

Main goals

- Prove a relation between c-Rayleigh and M-convex

Theorem 2.23. If f is homogeneous and c-Rayleigh, then the support of f is M-convex.

- Prove relations between c-Rayleigh and Lorentzian

Proposition 2.19. Any polynomial in L_n^d is $2\left(1-\frac{1}{d}\right)$-Rayleigh.

Proposition 2.24. When $n \leqslant 2$, all polynomials in L_n^d are 1-Rayleigh. When $n \geqslant 3$, we have

$$\left(\text{all polynomials in } \mathrm{L}_n^d \text{ are } c\text{-Rayleigh}\right) \Longrightarrow c \geqslant 2\left(1-\frac{1}{d}\right).$$

In other words, for any $n \geqslant 3$ and any $c < 2\left(1-\frac{1}{d}\right)$, there is $f \in \mathrm{L}_n^d$ that is not c-Rayleigh.

Preservers of c-Rayleigh polynomials

$\partial^\alpha f \cdot \partial^{\alpha+e_i+e_j} f \leq c \cdot \partial^{\alpha+e_i} f \cdot \partial^{\alpha+e_j} f$

Lemma 2.20. The following polynomials are c-Rayleigh whenever f is c-Rayleigh:

(1) The *contraction* $\partial_i f$ of f.

(2) The *deletion* $f \setminus i$ of f, the polynomial obtained from f by evaluating $w_i = 0$.

(3) The *diagonalization* $f(w_1, w_1, w_3, \ldots, w_n)$.

(4) The *dilation* $f(a_1 w_1, \ldots, a_n w_n)$, for $(a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$.

(5) The *translation* $f(a_1 + w_1, \ldots, a_n + w_n)$, for $(a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$.

$g(w_1, \ldots, w_n) = f(a_1 w_1, \ldots, a_n w_n)$

$\partial_i g = a_i \partial_i f$

Whiteboard

If f : multiaffine, then f being c-Rayleigh

$\iff \quad \forall i,j \in [n], \ f \cdot \partial_i \partial_j f \leq c \cdot \partial_i f \cdot \partial_j f$ on $\mathbb{R}^n_{\geq 0}$. $\quad (*)$

$\alpha \in \{0,1\}^n \qquad \underbrace{f = g + w^\alpha \cdot h}$

Want to show that $\underbrace{h \cdot \partial_i \partial_j h \leq c \cdot \partial_i h \cdot \partial_j h}$ for $i,j \in [n] \setminus \mathrm{supp}(\alpha)$

By $(*)$, $\quad (g + w^\alpha \cdot h) \cdot (\partial_i \partial_j g + w^\alpha \cdot \partial_i \partial_j h) \leq c \cdot (\partial_i g + w^\alpha \cdot \partial_i h) \cdot (\partial_j g + w^\alpha \cdot \partial_j h)$

Just let $w_i \to \infty$ for $i \in \alpha$.

$M^\natural$-convexity

- Recall M-convexity:

For any $\alpha, \beta \in J$ and $i \in [n]$ s.t. $\alpha_i > \beta_i$, there exists $j \in [n]$ s.t. $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$

- $J^\natural \subseteq \mathbb{N}^n$ is said to be $M^\natural$-convex if "$J^\natural$ is obtained from an M-convex $J \subseteq \mathbb{N}^{n+1}$ by deleting one coordinate"

$\{(2,0,0), (1,0,1), (1,1,0), (0,1,1)\}$: M-convex

$\{(2,0), (1,0), (1,1), (0,1)\}$: $M^\natural$-convex.

Key lemma

Lemma 2.22. Let f be a c-Rayleigh polynomial in $\mathbb{R}[w_1, \ldots, w_n]$.

(1) The support of f is interval convex.

(2) If $f(0)$ is nonzero, then $\operatorname{supp}(f)$ is $\mathrm{M}^\natural$-convex.

- Interval convex: If $\alpha, \beta \in J$ and $\alpha \leq \gamma \leq \beta$, then $\gamma \in J$

(if $\alpha \leq \beta$)

Think of $J = \operatorname{supp}(f)$.

Lemma 2.22. Let f be a c-Rayleigh polynomial in $\mathbb{R}[w_1, \ldots, w_n]$.

(1) The support of f is interval convex.

(2) If $f(0)$ is nonzero, then $\mathrm{supp}(f)$ is $M^\natural$-convex.

Whiteboard

f : c-Rayleigh then $J = \mathrm{supp}(f)$ is interval convex.

Suppose not. $\Rightarrow \exists \alpha, \beta \in J$ s.t. $\alpha < \beta$ and $\exists \gamma$ s.t. $\alpha \leq \gamma \leq \beta$ but $\gamma \notin J$.

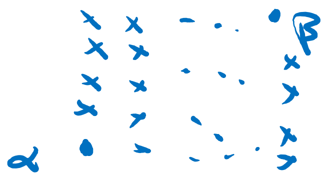
Choose so that $|\beta - \alpha|_1$ is minimized.

In this case: $\forall \gamma$ s.t. $\alpha \leq \gamma \leq \beta$ and $\gamma \neq \alpha, \beta \Rightarrow \gamma \notin J$.

Originally $f = c_\alpha w^\alpha + c_\beta w^\beta + \ldots$

Apply ∂_α, rescale $\Rightarrow$

$f = 1 + c_{\beta'} w^{\beta'} + \ldots$



$f \cdot \underline{\partial_i \partial_j f}$ $\quad c \cdot \partial_i f \cdot \partial_j f$

$1 \cdot (w^{\beta' - e_i - e_j} + \ldots)$ $\quad (w^{\beta' - e_i} + \ldots)(w^{\beta' - e_j} + \ldots)$

$O(w^{|\beta'| - 2})$ $\quad O(w^{2|\beta'| - 2})$

LHS $\gg$ RHS as $w \to 0$.

Lemma 2.22. Let f be a c-Rayleigh polynomial in $\mathbb{R}[w_1, \ldots, w_n]$.

(1) The support of f is interval convex.

(2) If $f(0)$ is nonzero, then $\mathrm{supp}(f)$ is $M^\natural$-convex.

Whiteboard

(2) $f(0) \neq 0$ $(0 \in \mathrm{supp}(f)) \Rightarrow \mathrm{supp}(f)$ is $M^\natural$-convex

Lemma (2.21) J: interval convex, $0 \in J$. Then J is $M^\natural$-convex iff J satisfies the "augmentation property".

$\left(\alpha, \beta \in J,\ |\alpha|_1 < |\beta|_1\right.$, then you can find $i \in [n]$, s.t. $\left.\alpha_i < \beta_i \text{ and } \alpha + e_i \in J. \right)$

Lemma 2.22. Let f be a c-Rayleigh polynomial in $\mathbb{R}[w_1, \dots, w_n]$.

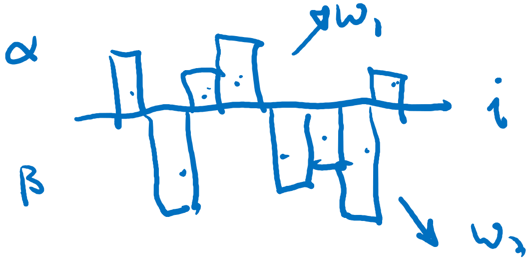
(1) The support of f is interval convex.

(2) If $f(0)$ is nonzero, then $\mathrm{supp}(f)$ is $\mathrm{M}^\natural$-convex.

Whiteboard

$|\alpha|_1 < |\beta|_1$, $\alpha_i < \beta_i \Rightarrow \alpha + e_i \notin J$.

Assume that $\{i : \alpha_i > 0\}$ and $\{i : \beta_i > 0\}$ are disjoint.



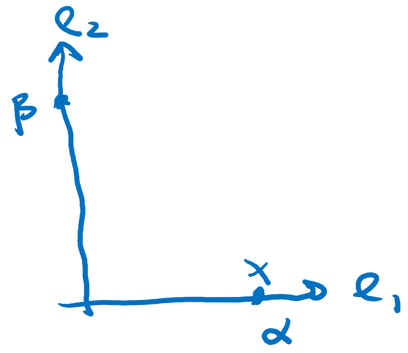
Target: find $\gamma \in J$ s.t.

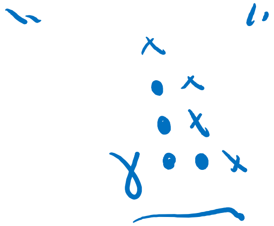
$\gamma + e_1 \in J$

$\gamma + e_2 \in J$

$\gamma + 2e_2 \in J$

$\gamma + e_1 + e_2 \notin J$.

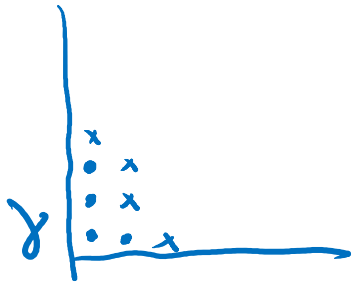




Look at $h(k)$: the smallest number s.t. $k \cdot e_2 + h(k) \cdot e_1 \notin J$

- $h(0) > |\beta| \geq |\alpha| \Rightarrow h(0) \geq |\alpha| + 2$
- $h(|\alpha|) = 1$

Recap



∂^γ to reduce to this.

Then check that this violates C-Rayleigh conditions.

$$i = j = 2$$

Theorem 2.23. If f is homogeneous and c-Rayleigh, then the support of f is M-convex.

Proof flow of Theorem 2.23

$g(w_1, \dots, w_n) = f(w_1+1, w_2+1, \dots, w_n+1)$

g is c-Rayleigh ✓

const term of g is nonzero ✓

$\Rightarrow$ supp(g) is $M^{\natural}$-convex.

supp(f) is just the degree-d slice of supp(g)

($\because$ f is homogeneous)

$\therefore$ supp(f) is M-convex.

c-Rayleigh and Lorentzian

Proposition 2.19. Any polynomial in L_n^d is $2\left(1-\frac{1}{d}\right)$-Rayleigh. HR + Euler's identities

Proposition 2.24. When $n \leqslant 2$, all polynomials in L_n^d are 1-Rayleigh. When $n \geqslant 3$, we have

$$\left(\text{all polynomials in } L_n^d \text{ are } c\text{-Rayleigh}\right) \Longrightarrow c \geqslant 2\left(1-\frac{1}{d}\right).$$

In other words, for any $n \geqslant 3$ and any $c < 2\left(1-\frac{1}{d}\right)$, there is $f \in L_n^d$ that is not c-Rayleigh.

$$f = 2\left(1-\frac{1}{d}\right) w_1^d + w_1^{d-1} w_2 + w_1^{d-1} w_3 + w_1^{d-2} w_2 w_3 .$$

Whiteboard

$f \in L_n^d$ Lorentzian $\Rightarrow$ f is $2(1-\frac{1}{d})$-Rayleigh.

Only need to use "H_f has ≤ 1 positive eigenvalue".

Outline: use Euler's identity to get

- $w^T H_f w = d(d-1) f \quad (\in \mathbb{R})$
- $w^T H_f = (d-1) \nabla f \quad (\in \mathbb{R}^n)$

$\Rightarrow w^T H_f e_i = (d-1) \partial_i f$

- $e_i^T H_f e_j = \partial_i \partial_j f$

Plug these in, consider $\begin{bmatrix} -(e_i + te_j)^T- \\ -w- \end{bmatrix} H_f \begin{bmatrix} | & | \\ e_i+te_j & w \\ | & | \end{bmatrix}$, t: real parameter

determinant ≤ 0 will give you the correct c-Rayleigh condition.

Agenda

- c-Rayleigh property
- Generating polynomial of M-convex sets
- CLC $\Leftrightarrow$ Lorentzian
- Proof of Mason's conjecture

Let $f_J := \sum_{\alpha \in J} \frac{w^\alpha}{\alpha!}$. Then: $J \subseteq \mathbb{N}^n$

Special case — $J \subseteq \{0,1\}^n$. $f_J = \sum_{\alpha \in J} w^\alpha$.

Theorem 3.10. The following are equivalent for any nonempty $J \subseteq \mathbb{N}^n$.

(1) There is a Lorentzian polynomial whose support is J.

(2) There is a homogeneous 2-Rayleigh polynomial whose support is J.

(3) There is a homogeneous c-Rayleigh polynomial whose support is J for some $c > 0$.

(4) The generating function f_J is a Lorentzian polynomial.

(5) The generating function f_J is a homogeneous 2-Rayleigh polynomial.

(6) The generating function f_J is a homogeneous c-Rayleigh polynomial for some $c > 0$.

(7) J is M-convex.

When $J \subseteq \{0,1\}^n$, any one of the above conditions is equivalent to

(8) J is the set of bases of a matroid on $[n]$.

Whiteboard

$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$

$(4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7)$

$$\begin{tikzcd}
(1) \arrow[rr, Rightarrow, bend left] & (4) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (7) \arrow[l, Rightarrow, bend left, "\text{(soon)}"]
\end{tikzcd}$$

(1) There exists $f \in L^d_n$ with $\mathrm{supp}(f) = J$

(4) $f_J \in L^d_n$

(7) J is M-convex.

Need to check:

$\mathrm{supp}(f_J)$ is M-convex.

$\partial^\alpha f_J$ is Lorentzian for any $|\alpha| = d-2$.

Whiteboard

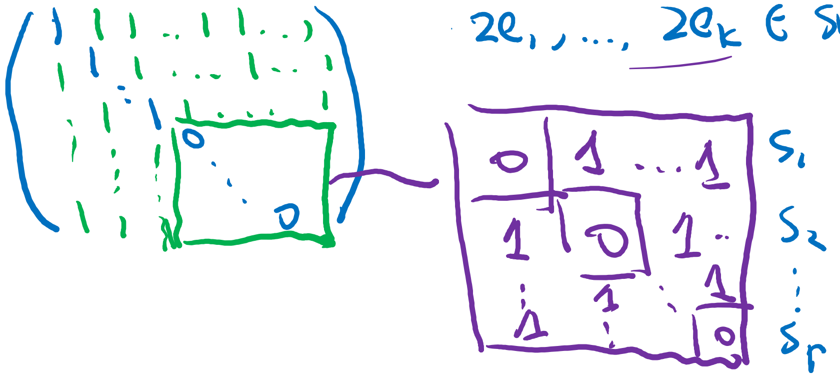
Let $\underline{A \in \{0,1\}^{n\times n}}$, symmetric. Then the quadratic form

$w^T A w$ is M-convex iff it is Lorentzian.

Really need $\underline{\text{M-convex}} \Rightarrow$ Lorentzian.

(Remove zero rows and columns)

$\underline{2e_1, \dots, 2e_k} \in \operatorname{supp}(w^T A w)$

$A =$ 

$$A = \mathbb{1}\mathbb{1}^T - \mathbb{1}_{S_1}\mathbb{1}_{S_1}^T - \dots - \mathbb{1}_{S_r}\mathbb{1}_{S_r}^T$$, certainly has ≤ 1 positive e-value.

Significance

- Lorentzian polynomials have M-convex supports
  - Generalized from real stable polynomials
- Generating polynomial of M-convex set is Lorentzian
  - Generalized the result proved in *log-concave-i* [AOV '18]
- Conjecture 3.12: better constant than 2 for matroids?

Agenda

- c-Rayleigh property
- Generating polynomial of M-convex sets
- CLC $\Leftrightarrow$ Lorentzian
- Proof of Mason's conjecture

Equivalence of CLC and Lorentzian

- Now we have enough tools to prove:

Theorem 2.30. The following conditions are equivalent for any homogeneous polynomial f.

(1) f is completely log-concave.

(2) f is strongly log-concave.

(3) f is Lorentzian.

Relating different Hessians

Proposition 2.33. The following are equivalent for any $w \in \mathbb{R}^n$ satisfying $f(w) > 0$.

(1) The Hessian of $f^{1/d}$ is negative semidefinite at w.

(2) The Hessian of $\log f$ is negative semidefinite at w.

(3) The Hessian of f has exactly one positive eigenvalue at w.

SLC $\Rightarrow$ Lorentzian

- Let $f \in H_n^d$ be SLC and nonzero

- Then, $H_{\partial^\alpha f}$ has exactly one positive eigenvalue on $\mathbb{R}^n_{>0}$

$\begin{pmatrix} - & u^T & - \\ - & v^T & - \end{pmatrix} H \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$ $\quad f \cdot \partial_i \partial_j f \leq c \cdot \partial_i f \cdot \partial_j f$

- This means f is c-Rayleigh, and so $supp(f)$ is M-convex

(See "Lorentzian $\Rightarrow$ $2(1-\frac{1}{d})$-Rayleigh")

- Base case satisfied, so Lorentzian

Lorentzian $\Rightarrow$ CLC

- Let $f \in L_n^d$ be nonzero

- Hodge-Riemann + Prop. 2.33 $\Rightarrow f$ is log-concave on $\mathbb{R}_{>0}^n$

  consequence of $f(Aw)$

- By last time, $(\sum_i a_i \partial_i) f \in L_n^{d-1}$ for $a_i \geq 0$

- This proves that partials of f are log-concave, so f is CLC

A Consequence

Corollary 2.32. The product of strongly log-concave homogeneous polynomials is strongly log-concave.

$f \in L_n^{d_1}$, $g \in L_n^{d_2}$

$f(w_1, \dots, w_n) \cdot g(w_{n+1}, \dots, w_{2n}) \in L_{2n}^{d_1+d_2}$

Then diagonalize.

Agenda

- c-Rayleigh property
- Genrating polynomial of M-convex sets
- CLC $\Leftrightarrow$ Lorentzian
- Proof of Mason's conjecture

Summary

Note: Mason's conjecture deferred to next week.

I added some proof outlines (in red) for reference.

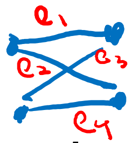
Some loose ends…

- Give an example of $f \in H_n^d$ that is log-concave but not CLC
  - What if we require $f = f_J$ for some $J \subseteq \mathbb{N}^n$?
  - What if we require $f = f_J$ for some $J \subseteq \{0,1\}^n$?
  - What if we require $n = 2$, i.e. that f be bivariate?

- Are intersections of M-convex sets convex? $\boxed{\text{No}}$

$=$ $\times$

$J_1 \cap J_2$ $= \{(1,0,0,1), (0,1,1,0)\}$.



$J_1 = \{(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1)\}$

$J_2 = \{(1,1,0,0), (0,1,1,0), (1,0,0,1), (0,0,1,1)\}$

- How to prove that diagonalization preserves c-Rayleigh property?

$\partial_1 f \cdot \partial_2 \partial_i \partial_j f \le c \cdot \partial_1 \partial_i f \cdot \partial_2 \partial_j f$ $\quad g \cdot \partial_i \partial_j g \quad g = \partial^\alpha f$

$(\partial_1 + \partial_2) f \cdot (\partial_1 + \partial_2) \partial_i \partial_j f$

Next time:

- Proof of Mason’s conjecture
- Preservers of Lorentzian polynomials