Reading Group W21
Lorentzian Polynomials (Part I)

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Agenda

- Definition and examples
- Relating it to other polynomial families
- M-convexity and support of Lorentzian polynomials
- Hodge-Riemann relation

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Definition

- H_n^d: polynomials in $\mathbb{R}[w_1, \ldots, w_n]$ of homogeneous degree d
- $P_n^d \subset H_n^d$: those with **positive** coefficients

Strictly Lorentzian polynomials:

- $\dot{L}_n^2 :=\ \{f \in P_n^2 : H_f$ has Lorentzian signature $(+, -, \ldots, -)\}$
- $\dot{L}_n^d :=\ \{f \in P_n^d : \partial_i f \in \dot{L}_n^{d-1}$ for any $i \in [n]\}$

Lorentzian polynomials: take **limit**

Example: Bivariate polynomials

- Consider $f = w_1^3 + w_2^3$
  - $\partial_1 f = 3w_1^2, \partial_2 f = 3w_2^2$ are Lorentzian (check Hessian)…
  - But f is not!

- In fact, $f = \sum_{i=0}^{d} a_i w_1^i w_2^{d-i}$ is strictly Lorentzian iff the sequence (a_i) is strictly ultra log-concave

$$\left(\frac{a_i}{\binom{d}{i}}\right)^2 > \frac{a_{i-1}}{\binom{d}{i-1}} \cdot \frac{a_{i+1}}{\binom{d}{i+1}}$$

Example: Quadratics

Lemma 2.5. The following conditions are equivalent for any $f \in \mathrm{P}_n^2$.

(1) The Hessian of f has the Lorentzian signature $(+, -, \ldots, -)$, that is, $f \in \mathring{\mathrm{L}}_n^2$.

(2) For any nonzero $u \in \mathbb{R}_{\geqslant 0}^n$, $(u^T \mathcal{H}_f v)^2 > (u^T \mathcal{H}_f u)(v^T \mathcal{H}_f v)$ for any $v \in \mathbb{R}^n$ not parallel to u.

(3) For some $u \in \mathbb{R}_{\geqslant 0}^n$, $(u^T \mathcal{H}_f v)^2 > (u^T \mathcal{H}_f u)(v^T \mathcal{H}_f v)$ for any $v \in \mathbb{R}^n$ not parallel to u.

(4) For any nonzero $u \in \mathbb{R}_{\geqslant 0}^n$, the univariate polynomial $f(xu - v)$ in x has two distinct real zeros for any $v \in \mathbb{R}^n$ not parallel to u.

(5) For some $u \in \mathbb{R}_{\geqslant 0}^n$, the univariate polynomial $f(xu - v)$ in x has two distinct real zeros for any $v \in \mathbb{R}^n$ not parallel to u.

Whiteboard

(1) $\Rightarrow$ (2)

H_f has signature $(+, -, \ldots, -)$

What we need to prove:

$\exists u \geq 0,\ u \neq \vec{0}$, s.t. $(u^T H_f v)^2 > (u^T H_f u) \cdot (v^T H_f v)$

for any v (not $\parallel$ to u.)

$$\begin{pmatrix} - u^T - \\ - v^T - \end{pmatrix} \begin{pmatrix} H_f \end{pmatrix} \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix} = \begin{pmatrix} u^T H_f u & u^T H_f v \\ v^T H_f u & v^T H_f v \end{pmatrix}$$

H_f projected on span$(\{u, v\})$ has signature $(+, -)$

$\Rightarrow$ det is negative.

Whiteboard

(2) ⇒ (4) $\exists u \geq 0$, $u \neq \vec{0}$, s.t. the univariate poly $f(xu+v)$ has two distinct real roots for v not $\parallel$ to u.

$f(w) = \sum c_{ij} w_i w_j$

$$f(xu+v) = \sum_{i \le j} c_{ij}(xu_i + v_i)(xu_j + v_j)$$

$$= \sum_{i \le j} c_{ij}\left[x^2 u_i u_j + x(u_i v_j + u_j v_i) + v_i v_j\right]$$

f has 2 distinct real roots $\Leftrightarrow \Delta > 0$

$$\Leftrightarrow (u^T H_f v)^2 > (u^T H_f u)(v^T H_f v)$$

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Real Stable Polynomials

- $S_n^d \subset H_n^d$: *real stable* (= nonvanishing on $\mathbb{H}^n$) with nonnegative coefficients

Examples:

- $\det(A + w_1 B_1 + \cdots + w_n B_n)$ where $B_i \succcurlyeq 0$
- Spanning tree polynomial: $\sum_T (\prod_{e \in T} w_e)$

Properties

- $f \in H_n^d$ is real stable iff: for any $u, v \in \mathbb{R}^n$ with $u \geq 0$ and $f(u) > 0$,
  $$f(xu + v) \text{ is real-rooted as a polynomial in } \mathbb{R}[x]$$
- Some preservers of real stability:
  - Partial derivative
  - Product
  - $\mathbb{R}$-specialization
  - Projection
  - Inversion
  - …

Whiteboard

Proposition 2.2. Any polynomial in S_n^d is Lorentzian.

- $d = 2$: comparing the statements

strictly Lorentzian $\Leftrightarrow$ $u \geq 0$, $u \neq \vec{0}$, then $f(xu+v)$ has two distinct real roots.

quadratic real stable $\Leftrightarrow$ $u \geq 0$, $u \neq \vec{0}$, then $f(xu+v)$ is real rooted.

- $d > 2$: use the fact that ∂_i preserves (strict) real-stability

Consequence

- Real stable polynomials form an important and well-studied subclass of Lorentzian polynomials :)

Completely Log-Concave Polynomials

- $f \in H_n^d$ is *log-concave* if f has nonnegative coefficients and:
  - either $f \equiv 0$, or
  - $H_{\log f}|_{w=x} \preccurlyeq 0$ for any $x \in \mathbb{R}_{>0}^n$

- $f \in H_n^d$ is *strongly log-concave* if $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} f$ is log-concave for any sequence of partial derivatives of length $0 \leq k \leq d-2$

- Completely log-concave: replace each ∂_i by $\sum_j a_{ij}\partial_j$ for $a_{ij} \geq 0$

Theorem 2.30. The following conditions are equivalent for any homogeneous polynomial f.

(1) f is completely log-concave.

(2) f is strongly log-concave.

(3) f is Lorentzian.

- Proof will be deferred
- Things we know about CLC polynomials (from *log-concave-{i, ii, iii, iv}*) apply to Lorentzian polynomials!

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Remedy for the $f = w_1^3 + w_2^3$ example

- Two ways to look at a polynomial $f \in \mathbb{R}[w_1, w_2, \ldots, w_n]$

Function

- $f : \mathbb{F}^n \to \mathbb{F}$
- $(w_1, \ldots, w_n) \mapsto f(w_1, \ldots, w_n)$

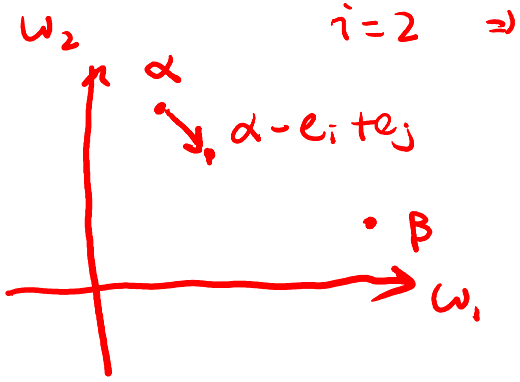
Coefficients

- $f = \sum_{\alpha \in \mathbb{N}^n} \frac{c_\alpha}{\alpha!} w^\alpha$
- $c : \alpha \mapsto c_\alpha$

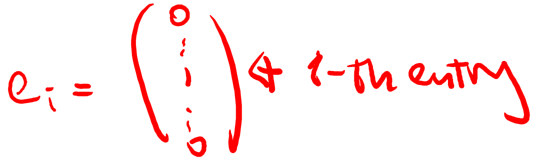
"ultra log-concavity of coefficients of bivariates
⇝ log-concave in normalized coeffs.

- We shall impose a condition on $supp(f)$

M-convexity



- Let $J \subseteq \mathbb{N}^n$
- J is M-convex iff:
  - For any $\alpha, \beta \in J$ and $i \in [n]$ such that $\alpha_i > \beta_i$, there is $j \in [n]$ such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$



- This is called *exchange property*

- M_n^d: polynomials in H_n^d with non-negative coefficients, whose support is M-convex

Basic properties

Let J be M-convex. $\alpha, \beta \in J$, $\sum \alpha_i = \sum \beta_i$

- Each element in J has the same (l_1-)sum
- If $J \subseteq \{0,1\}^n$, then J is the bases of a matroid
- Support of $f \in S_n^d$ is M-convex [Bränden 07]

$[l_1, r_1] \times [l_2, r_2] \times \cdots \times [l_n, r_n]$

- Intersection of J with a rectangle is again M-convex
- Splitting and aggregation preserve M-convexity

Whiteboard

(1) Each $\alpha \in J$ has the same l_1-sum.

Find $\alpha, \beta \in J$, $|\alpha|_1 < |\beta|_1$.

Target: end up with α', $|\alpha|_1 = |\alpha'|_1$, $\alpha' \leq \beta$

Whenever there's $i \in [n]$ s.t. $\alpha_i > \beta_i$,

find corresponding $j \in [n]$ s.t. $\alpha_j < \beta_j$ and

$\underbrace{\alpha - e_i + e_j}$ $\in J$.

Replace α by $\alpha - e_i + e_j$, repeat.

Whiteboard

$f(w_1, \dots, w_n) \longrightarrow f(w_1, \dots, w_{n-1}, (w_n + w_{n+1}))$

Splitting: $J \to J'$

$(\alpha_1, \dots, \alpha_n) \Rightarrow (\alpha_1, \dots, \alpha_{n-1}, \alpha_n', \alpha_{n+1}')$ s.t. $\alpha_n' + \alpha_{n+1}' = \alpha_n$.

Aggregation: $J \to J''$

$(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \Rightarrow (\alpha_1, \dots, \alpha_{n-1}, \alpha_n + \alpha_{n+1})$

$\alpha' = (\alpha_1, \dots, \alpha_{n-1}, \alpha_n', \alpha_{n+1}') \in J'$, suppose $\alpha_i' > \beta_i'$

$\beta' = (\beta_1, \dots, \beta_{n-1}, \beta_n', \beta_{n+1}')$

Case 1: $i \in [n-1]$. Look at $\alpha, \beta \in J$. Find $j \in [n]$, s.t. $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$. (What if $j = n$? Pick one of n or $n+1$.)

Case 2: $i = n$ ($i = n+1$ is similar). Potentially $\alpha_n' > \beta_n'$ but $\alpha_n' + \alpha_{n+1}' \leq \beta_n' + \beta_{n+1}'$. Pick $j = n+1$. $\alpha' - e_n + e_{n+1} \in J'$

Note

- Intersection of two M-convex sets may not be convex!

$J_1 = \{(2,0,0), (1,1,0), (0,1,1)\}$

$J_2 = \{(2,0,0), (1,0,1), (0,1,1)\}$

$J_1 \cap J_2 = \{(2,0,0), (0,1,1)\}$, not M-convex!

α, β : look at $[m_1, M_1] \times [m_2, M_2] \times \dots \times [m_n, M_n]$.

$m_i := \min(\alpha_i, \beta_i)$

$M_i := \max(\alpha_i, \beta_i)$

Defining L_n^d

- For $d \le 2$, $L_n^d := S_n^d$
- For $d > 2$, $L_n^d := \{f \in M_n^d : \partial_i f \in L_n^{d-1} \text{ for any } i \in [n]\}$

- Alternatively, for any $\partial_{i_1} \dots \partial_{i_k} f$, it is in M_n^{d-k}.

$$L_n^d := \{f \in M_n^d : \partial_{i_1} \cdots \partial_{i_{d-2}} f \in L_n^2 \text{ for any } i_1, \ldots i_{d-2} \in [n]\}$$

$f = w_1^3 + w_2^3 \notin L_n^d$

$\{(3,0), (0,3)\}$ not M-convex

Key facts about L_n^d

Theorem 2.10. If $f(w) \in \mathrm{L}_n^d$, then $f(Av) \in \mathrm{L}_m^d$ for any $n \times m$ matrix A with nonnegative entries.

- Rather general class of operators that preserve L_n^d

Theorem 2.25. The closure of $\mathring{\mathrm{L}}_n^d$ in H_n^d is L_n^d. In particular, L_n^d is a closed subset of H_n^d.

- L_n^d = Lorentzian!

Proof (sktech) of Theorem 2.10

- Prove that $f \in L_n^d \Rightarrow (1+\theta w_i \partial_j) f \in L_n^d$ for any $\theta \geq 0$

- Suffices to consider the following elementary operations:
  - Elementary splitting $f(w_1, \dots, w_{n-1}, w_n + w_{n+1}) \in L_{n+1}^d$
  - Dilation $f(w_1, \dots, w_{n-1}, \theta w_n) \in L_n^d$ for $\theta \geq 0$
  - Diagonalization $f(w_1, \dots, w_{n-2}, w_{n-1}, w_{n-1}) \in L_{n-1}^d$

Whiteboard

Weapon: $f \in L_n^d$, $(1+\theta w_i \partial_i) f \in L_n^d$.

(1) Splitting. $f(w_1,\dots,w_n) \to f(w_1,\dots,w_n+w_{n+1})$.

$$\lim_{k\to\infty} \left(1+\frac{w_{n+1}\partial_n}{k}\right)^k f = f(w_1,\dots,w_n+w_{n+1})$$

$f = w_n^\ell$. RHS we get $(w_n+w_{n+1})^\ell$.

Fix k.

$$\text{LHS} = \left(1+\frac{w_{n+1}\partial_n}{k}\right)^k \cdot w_n^\ell = \sum_{i=0}^{k} \binom{k}{i} \cdot \left(\frac{w_{n+1}\partial_n}{k}\right)^i \cdot w_n^\ell$$

$$\binom{k}{i} = \frac{k(k-1)\cdots(k-i+1)}{i!}$$

for large k $= \sum_{i=0}^{\ell} \binom{k}{i} \cdot \frac{w_{n+1}^i w_n^{\ell-i}}{k^i} \cdot \ell \cdot \dots \cdot (\ell-i+1)$

$$\xrightarrow{k\to\infty} = \sum_{i=0}^{\ell} \binom{\ell}{i} \cdot w_{n+1}^i \cdot w_n^{\ell-i}.$$

Whiteboard

(2) Dilation $\quad f(w_1, \ldots, w_n) \to f(w_1, \ldots, \theta w_n)$

(3) Diagonalization $\quad f(w_1, \ldots, w_n) \to f(w_1, \ldots, w_{n-1}, \underline{w_{n-1}})$

Why general case holds

$2 = n$ vars $\qquad\qquad$ $m = 3$ vars

$$f(w_1, w_2) \longrightarrow f(z_1 + z_2,\ 2z_1 + \tfrac{1}{2} z_3)$$

$\uparrow z_{11}, z_{12}, z_{13} \qquad \uparrow z_{21}, z_{22}, z_{23}$

$\downarrow$ Splitting nm times

$$f(w_1 + z_{11} + z_{12} + z_{13},\ w_2 + z_{21} + z_{22} + z_{23})$$

$\downarrow$ Dilation

$$f(0 \cdot w_1 + 1 \cdot z_{11} + 1 \cdot z_{12} + 0 \cdot z_{13},\ 0 \cdot w_2 + 2 \cdot z_{21} + 0 \cdot z_{22} + \tfrac{1}{2} z_{23})$$

$$= f(z_{11} + z_{12},\ 2z_{21} + \tfrac{1}{2} z_{23})$$

$\downarrow$ Diagonalization : identify z_{ij} and $z_{i'j}$.

$$f(z_1 + z_2,\ 2z_1 + \tfrac{1}{2} z_3).$$

An important consequence

Corollary 2.11. If $f \in \mathrm{L}_n^d$, then $\left(\sum_{i=1}^n a_i \partial_i\right) f \in \mathrm{L}_n^{d-1}$ for any $a_1, \ldots, a_n \geqslant 0$.

- Note that $f(w_1 + a_1 w_{n+1}, \ldots, w_n + a_n w_{n+1}) \in L_{n+1}^d$
- Apply ∂_{n+1}, then set $w_{n+1} = 0$

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Hodge-Riemann relation

- Meaning "negative semidefinite on a dimension-(n−1) subspace"
- We want to prove the following statement:

Theorem 2.16. Let f be a nonzero homogeneous polynomial in $\mathbb{R}[w_1, \ldots, w_n]$ of degree $d \geqslant 2$.

(1) If f is in $\mathring{\mathrm{L}}_n^d$, then $\mathcal{H}_f(w)$ is nonsingular for all $w \in \mathbb{R}^n_{>0}$.

(2) If f is in L_n^d, then $\mathcal{H}_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}^n_{>0}$.

Theorem 2.16. Let f be a nonzero homogeneous polynomial in $\mathbb{R}[w_1, \ldots, w_n]$ of degree $d \geqslant 2$.

(1) If f is in $\mathring{\mathrm{L}}_n^d$, then $\mathcal{H}_f(w)$ is nonsingular for all $w \in \mathbb{R}_{>0}^n$.

(2) If f is in L_n^d, then $\mathcal{H}_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}_{>0}^n$.

Proof Strategy

Use induction on d to prove (1) + (2) simultaneously

- Step 1: Prove (1) first
- Step 2: Show that (2) holds for *some* polynomials in $\dot{L}_n^d$
- Step 3: Use (1) + connectedness of $\dot{L}_n^d$ to conclude (2)

Step 1

Theorem 2.16. Let f be a nonzero homogeneous polynomial in $\mathbb{R}[w_1, \ldots, w_n]$ of degree $d \geqslant 2$.

(1) If f is in $\mathring{\mathrm{L}}_n^d$, then $\mathcal{H}_f(w)$ is nonsingular for all $w \in \mathbb{R}^n_{>0}$.

(2) If f is in L_n^d, then $\mathcal{H}_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}^n_{>0}$.

Lemma 2.15. If $\mathcal{H}_{\partial_i f}(w)$ has exactly one positive eigenvalue for every $i \in [n]$ and $w \in \mathbb{R}^n_{>0}$, then

$$\ker \mathcal{H}_f(w) = \bigcap_{i=1}^{n} \ker \mathcal{H}_{\partial_i f}(w) \quad \text{for every } w \in \mathbb{R}^n_{>0}.$$

Whiteboard

Step 2

Theorem 2.16. Let f be a nonzero homogeneous polynomial in $\mathbb{R}[w_1, \dots, w_n]$ of degree $d \geqslant 2$.

(1) If f is in $\mathring{\mathrm{L}}^d_n$, then $\mathcal{H}_f(w)$ is nonsingular for all $w \in \mathbb{R}^n_{>0}$.

(2) If f is in L^d_n, then $\mathcal{H}_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}^n_{>0}$.

Proposition 2.14. If f is in $\mathrm{S}^d_n \backslash 0$, then $\mathcal{H}_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}^n_{>0}$. Moreover, if f is in the interior of S^d_n, then $\mathcal{H}_f(w)$ is nonsingular for all $w \in \mathbb{R}^n_{>0}$.

Whiteboard

(Path-)connectedness of $\dot{L}_n^d$

- For any $f \in \dot{L}_n^d \backslash \{0\}$, define

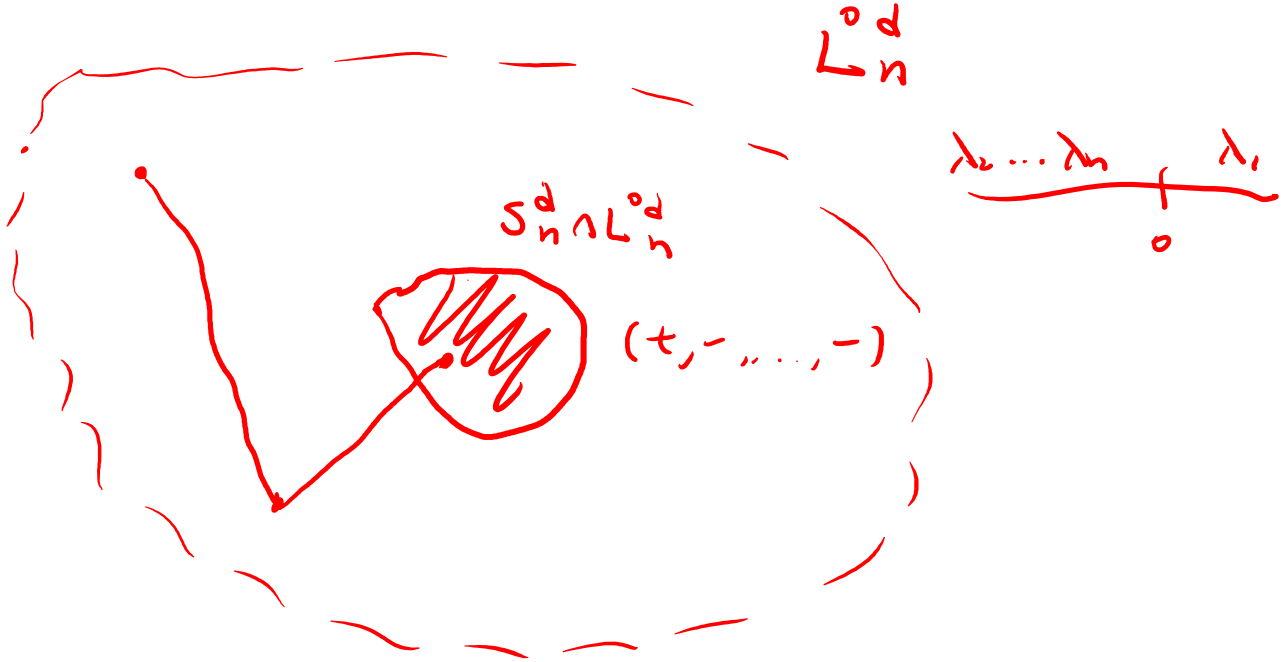
  $S(\theta, f) \coloneqq \frac{1}{|f|_1} f((1-\theta)w_1 + \theta(\sum_i w_i), \dots, (1-\theta)w_n + \theta(\sum_i w_i))$

- Define the operator $T_n(\theta, f) = (\prod_{i=1}^{n-1}(1 + \theta w_i \partial_n)^d) f$

- Then $T_n\big(\theta, S(\theta, f)\big) \in \dot{L}_n^d$ and deforms f continuously to $T_n(1, (\sum_i w_i)^d)$, as θ goes from 0 to 1

Concluding the proof

- Remains to prove (2) in the inductive step



Remark

- A similar statement is proved in *log-concave-iii* [ALOV '18]

Lemma 2.1. *Let $f \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree $d \geq 2$ with nonnegative coefficients. Fix a point $a \in \mathbb{R}^n_{\geq 0}$ with $f(a) \neq 0$, and let $Q = \nabla^2 f\big|_{z=a}$. The following are equivalent:*

(1) *f is log-concave at $z = a$,*

(2) *$z \mapsto z^\intercal Q z$ is negative semidefinite on $(Qa)^\perp$,*

(3) *$z \mapsto z^\intercal Q z$ is negative semidefinite on $(Qb)^\perp$ for every $b \in \mathbb{R}^n_{\geq 0}$ such that $Qb \neq 0$,*

(4) *$z \mapsto z^\intercal Q z$ is negative semidefinite on some linear space of dimension $n-1$, and*

(5) *the matrix $(a^\intercal Q a)Q - (Qa)(Qa)^\intercal$ is negative semidefinite.*

For $d \geq 3$, these are also equivalent to the condition

(6) *$D_a f$ is log-concave at $z = a$.*

Summary

Plan for Part II

- Negative dependence, Mason's conjecture
- Proof of CLC $\Leftrightarrow$ Lorentzian
- Generating functions of (discrete) convex sets
- Operations preserving Lorentzian polynomials