Log-Concave Polynomials I

Week 5: Reading Group W20

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Agenda

- Generating polynomial $g_M(z)$ is CLC
- CLC distributions and entropy
- Max. entropy programs and basis counting

I. Generating polynomial $g_M(z)$ is CLC

- Chow ring, graded Möbius algebra, and polynomials
- From $\nabla^2 g_M(z)$ to $\nabla^2(\log g_M(z))$

Known fact about Chow ring

- Given (simple) matroid M, on $E = [n]$, rank r
- $\mathbb{A}^*(M)$: Chow ring over the nonempty proper *flats* F of M
- It's an algebra (≈ vector space + ring)
- Elements $\sum_F c_F x_F$, $c_F \in \mathbb{R}$, modulo two relations
  - $x_{F_1} x_{F_2} = 0$ if F_1, F_2 incomparable
  - $\sum_{i \in F} x_F = \sum_{j \in F} x_F$ for $i, j \in E$

- Graded using (homogeneous) degree of polynomial

The deg(·) map

$\alpha_i = \sum_{i \in F} x_F$, then $\alpha_i = \alpha_j = \alpha$

① For any flag $F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k$,

$\exists j \in [k]$

if $rk(F_j) > j$, then

$x_{F_1} x_{F_2} \dots x_{F_k} \alpha^{r-k} = 0$

② If $rk(F_j) = j$ for all $j \in [k]$,

then $x_{F_1} x_{F_2} \dots x_{F_k} \alpha^{r-k} = \alpha^r$.

$x_{\{i\}} \alpha^{r-1} = \alpha^r$

$\left(\sum_{F \ni i} x_F\right) \alpha^{r-1}$

Fact. There exists an isomorphism

$$deg: \mathbb{A}^r(M) \to \mathbb{R}$$

s.t. $\deg(x_{F_1} x_{F_2} \dots x_{F_r}) = 1$ for *any* (maximal) flag $F_1 \subset F_2 \subset \cdots \subset F_r$

Elements $\sum_F c_F x_F$, $c_F \in \mathbb{R}$, modulo:

- $x_{F_1} x_{F_2} = 0$ if F_1, F_2 incomparable
- $\sum_{i \in F} x_F = \sum_{j \in F} x_F$ for $i, j \in E$

[Flat F, $i \notin F$,

$x_F \cdot \alpha = x_F \cdot \left(\sum_{G \supseteq F \cup \{i\}} x_G\right)$]

① $j = r-1$ ✓

$j \in F_k \setminus F_{k-1}$

$x_{F_1} \dots x_{F_k} \alpha^{r-k} = x_{F_1}$

The $\deg(\cdot)$ map

Fact. There exists an isomorphism

$$deg\colon \mathbb{A}^r(M) \to \mathbb{R}$$

s.t. $\deg\left(x_{F_1} x_{F_2} \ldots x_{F_r}\right) = 1$ for *any* (maximal) flag $F_1 \subset F_2 \subset \cdots \subset F_r$

$y \in F_k \setminus F_{k-1}$

$$x_{F_1} \ldots x_{F_k} \alpha^{r-k} \stackrel{?}{=} \alpha^r$$

$$\alpha^r = x_{F_1} \ldots x_{F_{k-1}} \alpha^{r-k+1}$$

$$= x_{F_1} \ldots x_{F_{k-1}} \Big(\sum_{F \ni j} x_F \Big) \alpha^{r-k}$$

$$= x_{F_1} \ldots x_{F_{k-1}} x_{F_k} \alpha^{r-k}$$

Graded Möbius algebra

- $\mathbb{B}^*(M)$: over flats F, modulo the relations
  - $y_{F_1}y_{F_2} - y_{F_1 \vee F_2}$, for $rank(F_1) + rank(F_2) = rank(F_1 \vee F_2)$
  - $y_{F_1}y_{F_2}$, for $rank(F_1) + rank(F_2) > rank(F_1 \vee F_2)$
- For any basis I of a flat F, $y_F = \prod_{i \in I} y_i$

Relating basis counting and Chow ring

- Turns out $\mathbb{B}^*(M)$ can be embedded into $\mathbb{A}^*(M_0)$!
  - $y_i \mapsto \beta_i \coloneqq \sum_{i\in F, 0\notin F} x_F$, extend naturally
- For basis B of M, $\prod_{i\in B} y_i = 1$ ∂y_E, so $\deg(\prod_{i\in B} \beta_i) = 1$ (up to scaling)
- For non-basis B of size r, $\prod_{i\in B} y_i = 0$

- **Prop. 3.3**. $\sum_{i=1}^{n} v_i \beta_i$ corresponds to ∂_v
- **Theorem 3.4**. For any $V \in \mathbb{R}_{\geq 0}^{n\times k}$, $0 \leq k \leq r-2$, the Hessian of $D_V g_M(z)|_{z=\lambda}$ has at most one positive eigenvalue.

From $\nabla^2 g_M(z)$ to $\nabla^2(\log g_M(z))$

- Let M be a simple matroid.
- **Theorem 3.4**. For any $V \in \mathbb{R}_{\geq 0}^{n\times k}$, $0 \leq k \leq r-2$, the Hessian of $D_V g_M(z)|_{z=\lambda}$ has at most one positive eigenvalue.

Goal:

- For any $V \in \mathbb{R}_{\geq 0}^{n\times k}$, $0 \leq k \leq r-2$, the Hessian of $\log D_V g_M(z)|_{z=\lambda}$ is negative semidefinite.

Whiteboard

Goal:

- For any $V \in \mathbb{R}_{\geq 0}^{n \times k}$, $0 \leq k \leq r-2$, the Hessian of $\log D_V g_M(z)|_{z=\lambda}$ is negative semidefinite.

$h(z) = D_V g_M(z)$.

Known: $\nabla^2 h(z)$ has ≤ 1 positive-value

$$\nabla^2 \log h(x) = \frac{1}{h^2(z)}\left[\underline{h(z)\nabla^2 h(z)} - \underline{(\nabla h(z))}(\nabla h(z))^T\right]$$

Euler's identities:

f: degree d homogeneous poly,

$z_1^{d_1} z_2^{d_2} \dots z_n^{d_n}$ $\left(\begin{array}{l} z_i \partial_i(\dots) \\ = d_i(\dots) \end{array} \right)$

① $\langle \nabla f(z), z \rangle = d \cdot f(z)$

② $\nabla^2 f(z) \cdot z = (d-1) \cdot \nabla f(z)$

③ $z^T \underline{\nabla^2 f(z) \cdot z} = d(d-1) \cdot f(z)$

Whiteboard

Goal:

- For any $V \in \mathbb{R}^{n\times k}_{\geq 0}$, $0 \leq k \leq r-2$, the Hessian of $\log D_V g_M(z)|_{z=\lambda}$ is negative semidefinite.

$h(z)\, D^2 h(z) - (\nabla h(z))(\nabla h(z))^T$ — $(*)$

Evaluated at $z=\lambda \in \mathbb{R}^n_{>0}$.

Set $A = D^2 h(z)|_{z=\lambda}$, $v=\lambda$,

Then $(*)|_{z=\lambda}$

$$= \frac{1}{d(d-1)}(v^T A v)\cdot A - \frac{1}{(d-1)^2}(Av)(Av)^T.$$

Goal: Show that this is $\preceq 0$.

$$(v^T A v)\cdot A - t\cdot (Av)(Av)^T \preceq 0 \text{ for } t \geq 1.$$

Whiteboard

Goal:

- For any $V \in \mathbb{R}_{\geq 0}^{n\times k}$, $0 \leq k \leq r-2$, the Hessian of log $\boxed{D_V g_M}(z)|_{z=\lambda}$ is negative semidefinite.

Consider $u \in \mathbb{R}^n$.

$$u^T\left[(v^TAv)\cdot A - (Av)(Av)^T\right]u \leq 0.$$

$$\Leftrightarrow (v^TAv)\cdot(u^TAu) - (u^TAv)^2 \leq 0.$$

$$\Leftrightarrow \det\Big(\underbrace{\begin{bmatrix} u^T \\ v^T \end{bmatrix} A \begin{bmatrix} u & v \end{bmatrix}}_{\begin{bmatrix} u^TAu & u^TAv \\ u^TAv & \underline{v^TAv} \end{bmatrix}}\Big) \leq 0$$

① It has ≤ 1 positive e-value.

② It has ≥ 1 non-negative e-value

(Some) affine maps preserve CLC

- Let $A \in \mathbb{R}_{\geq 0}^{n\times m}, b \in \mathbb{R}_{\geq 0}^{n}$. If $g(z_1, \dots, z_n) \in \mathbb{R}[z_1, \dots, z_n]$ is CLC, then $g(Az + b) \in \mathbb{R}[y_1, \dots, y_m]$ is also CLC.

$g \circ T \qquad Tz = Az + b$

1. $g(T(\lambda z_1 + (1-\lambda) z_2)) = g(\lambda T(z_1) + (1-\lambda) T(z_2))$

$\geq g(T(z_1))^{\lambda} \cdot g(T(z_2))^{1-\lambda}$.

2. $D_v(g(T(z)))\big|_{z=\lambda} = D_{A\cdot v}(g(y))\big|_{y=T(\lambda)}$

$(v \geq 0)$

From simple matroids to general matroids

M : matroid (not necessarily simple)

"simplify"

$\hat{M}$: Simple matroid

Delete loops, identify // elements.

$g_{\hat{M}}$ is CLC. find T s.t. $g_M = g_{\hat{M}} \circ T$.

(M, Z) loops

0 //

0 //

T maps loops to 0.

maps parallel elements

to ~~the~~ representative

in $\tilde{Z}$.

End of Part I

II. CLC distributions and entropy

- Approximation of $H(\mu)$ by $\sum_i H(\mu_i)$
- $p \in P_M$ as a limit of $\lambda * \mu$

Approximation of $H(\mu)$ by $\sum_i H(\mu_i)$

$\mu_i = \sum_{S \ni i} \mu_S$

- Let $\mu: 2^{[n]} \to \mathbb{R}_{\geq 0}$
- Subadditivity of entropy -> $H(\mu) \leq \sum_i H(\mu_i)$
- **Theorem 5.2**. If μ is *log-concave*, then $H(\mu) \geq \sum_i \mu_i \log\left(\frac{1}{\mu_i}\right)$.

- **Theorem 5.2**. If μ is *log-concave*, then $H(\mu) \geq \sum_i \mu_i \log\left(\frac{1}{\mu_i}\right)$.

Whiteboard

$$g_\mu(z) := \sum_S \mu_S z^S$$

$$h(z_1, \dots, z_n) := \log g_\mu\left(\frac{z_1}{\mu_1}, \dots, \frac{z_n}{\mu_n}\right).$$

Then h is a concave function.

$z \sim 1_S$ according to μ.

Jensen $\Rightarrow \mathbb{E}(h(z)) \leq h(\mathbb{E}(z))$

RHS: $h(\mathbb{E}(z)) = h(\mu_1, \dots, \mu_n)$

$= \log g_\mu(1, 1, \dots, 1) = \log\left(\sum_S \mu_S\right) = 0.$

Whiteboard

- **Theorem 5.2**. If μ is *log-concave*, then $H(\mu) \geq \sum_i \mu_i \log\left(\frac{1}{\mu_i}\right)$.

$$h(z_1, \dots, z_n) = \log g_\mu\left(\frac{z_1}{\mu_1}, \dots, \frac{z_n}{\mu_n}\right)$$

$$\text{LHS} = \mathbb{E}(h(z))$$

$$= \sum_S \mu_S \cdot h(\mathbb{1}_S)$$

$$= \sum_S \mu_S \cdot \log\left(\sum_T \mu_T \cdot \left(\frac{z}{\mu}\right)^T \Big|_{\mathbb{1}_S}\right)$$

$$\geq \sum_S \mu_S \cdot \log\left(\mu_S \cdot \prod_{i \in S} \frac{1}{\mu_i}\right)$$

$$= \underbrace{\sum_S \mu_S \cdot \log \mu_S}_{-H(\mu)} + \underbrace{\sum_S \mu_S \cdot \sum_{i \in S} \log\left(\frac{1}{\mu_i}\right)}_{= \sum_i \log\left(\frac{1}{\mu_i}\right) \cdot \sum_{S \ni i} \mu_S}$$

Consequences

- **Corollary 5.7.**
  $\max\left(\frac{1}{2}\sum_i H(\mu_i), \sum_i H(\mu_i) - r\right) \leq H(\mu) \leq \sum_i H(\mu_i)$ for μ: uniform distribution on bases of M

$$0 = RHS \geq LHS \geq -H(\mu) + \sum_i \mu_i \log\left(\frac{1}{\mu_i}\right)$$

$$\Rightarrow H(\mu) \geq \sum_i \mu_i \log\left(\frac{1}{\mu_i}\right).$$

$$H(\mu) \geq \sum_i \left[\mu_i \log\left(\frac{1}{\mu_i}\right) + (1-\mu_i)\log\left(\frac{1}{1-\mu_i}\right)\right]$$

$$- \sum_i (1-\mu_i) \log\left(\frac{1}{1-\mu_i}\right)$$

$$\geq \sum_i H(\mu_i) - \sum_i \mu_i \quad \left(x \geq (1-x)\log\left(\frac{1}{1-x}\right) \text{ for } x \in [0,1]\right)$$

$$= \sum_i H(\mu_i) - r$$

μ^*: uniform dist. on bases of M^*.

$\mu_i^* = 1 - \mu_i$

$$H(\mu) + H(\mu^*) \geq \sum_i \mu_i \log\left(\frac{1}{\mu_i}\right) + \sum_i \mu_i^* \log\left(\frac{1}{\mu_i^*}\right) = \sum_i H(\mu_i)$$

$p \in P_M$ as a limit of $\lambda * \mu$

- P_M: matroid polytope, i.e. $P_M = conv(1_B : B \in B_M)$

Lemma 5.8. *Let M be a matroid on ground set $[n]$ and let p be a point in $\mathcal{P}_M$. Then there is a distribution $\tilde{\mu}$ supported on $\mathcal{B}_M$ with marginals p, i.e., $\tilde{\mu}_i = p_i$, such that both $\tilde{\mu}$ and $\tilde{\mu}^*$ are completely log-concave. Furthermore $\tilde{\mu}$ and $\tilde{\mu}^*$ can be obtained as the limit of external fields applied to μ and μ^*, where μ is the uniform distribution on $\mathcal{B}_M$.*

End of Part II

III. Max. entropy programs and basis counting

- Counting $|B_M|$
- Counting $|B_M \cap B_N|$

Counting $|B_M|$

- Let $\tau = \max(\sum_i H(p_i) \,|\, p = (p_1, \ldots, p_n) \in P_M)$.
- To estimate $|B_M|$, output $\beta = e^{\tau}$.
- Then, β satisfies the following estimation guarantee:
$$\max(\sqrt{\beta}, e^{-r}\beta) \leq |B_M| \leq \beta$$
- This is the same as saying
$$\max\left(\frac{\tau}{2}, \tau - r\right) \leq \log(|B_M|) \leq \tau$$

Whiteboard

- Let $\tau = \max(\sum_i H(p_i) \mid p = (p_1, \dots, p_n) \in P_M)$. Then, $\max\left(\frac{\tau}{2}, \tau - r\right) \le \log(|B_M|) \le \tau$

- μ: uniform dist. on B_M, then $\mu \in P_M$, and so

$$\log|B_M| = H(\mu) \le \sum_i H(\mu_i) \le \tau$$

- Given p: optimal, can find $\hat{\mu}$: CLC, s.t. • $\hat{\mu}_i = p_i$.
  • $\hat{\mu}$ supp. on B_M.

$$H(\mu) \ge H(\hat{\mu}) \ge \max\left(\frac{1}{2}\sum_{i=1}^{n} H(\hat{\mu}_i), \sum_{i=1}^{n} H(\hat{\mu}_i) - r\right)$$

$$= \max\left(\frac{\tau}{2}, \tau - r\right).$$

Counting $|B_M \cap B_N|$ rank-r.

- Let $\tau = \max(\sum_i H(p_i) | \, p = (p_1, \dots, p_n) \in P_M \cap P_N)$.
- To estimate $|B_M \cap B_N|$, output $\beta = e^{\tau}$.
- Then, β satisfies the following estimation guarantee:
$$e^{-O(r)}\beta \leq |B_M \cap B_N| \leq \beta$$
- This is the same as saying
$$\tau - O(r) \leq \log(|B_M \cap B_N|) \leq \tau$$

Setup

$g_M(y) \cdot g_N(z)$

- Consider the polynomial $g(y, z) \coloneqq g_M(y) \cdot g_{N^*}(z)$
- Then, $\left(\prod_i (\partial y_i + \partial z_i)\right) g(y, z) = |B_M \cap B_N|$

$\sum_S (\partial y_S)(\partial z_{[n] \setminus S}) g_M(y) \cdot g_{N^*}(z)$

$y^{T_y} z^{T_z}$

$S \subseteq T_y, \ [n] \setminus S \subseteq T_z$

$S \in B_M, \ [n] \setminus S \in B_{N^*}$

$S \in B_M \cap B_N.$

- Target is to *lower-bound* LHS

Statement of the main theorem

Theorem 7.1. *Let* $g \in \mathbb{R}[y_1, \ldots, y_n, z_1, \ldots, z_n]$ *be a completely log-concave multiaffine polynomial and* $p \in [0,1]^n$. *Then,*

$$\left(\prod_{i=1}^{n}(\partial_{y_i} + \partial_{z_i})\right) g(y,z)\Big|_{y=z=0} \geq \phi(p) \cdot \inf_{y,z \in \mathbb{R}^n_{>0}} \frac{g(y,z)}{y^p z^{1-p}},$$

where $\phi(p)$ *is independent of the polynomial* g, *and is given by the following expression:*

$$\phi(p) = \prod_{i=1}^{n}\left(p_i^{p_i} \cdot (1-p_i)^{1-p_i} \cdot \frac{1}{1+p_i(1-p_i)}\right).$$

Implication

Theorem 7.1. *Let $g \in \mathbb{R}[y_1, \ldots, y_n, z_1, \ldots, z_n]$ be a completely log-concave multiaffine polynomial and $p \in [0,1]^n$. Then,*

$$\left(\prod_{i=1}^{n}(\partial_{y_i} + \partial_{z_i})\right) g(y,z) \Big|_{y=z=0} \geq \phi(p) \cdot \inf_{y,z \in \mathbb{R}^n_{>0}} \frac{g(y,z)}{y^p z^{1-p}},$$

where $\phi(p)$ is independent of the polynomial g, and is given by the following expression:

$$\phi(p) = \prod_{i=1}^{n}\left(p_i^{p_i} \cdot (1-p_i)^{1-p_i} \cdot \frac{1}{1+p_i(1-p_i)}\right).$$

Take log, plug in $g = g_M \cdot g_{N^*}$,

$$\log\left(\inf_{y>0} \frac{g_M(y)}{y^p}\right) + \log\left(\inf_{z>0} \frac{g_{N^*}(z)}{z^{1-p}}\right)$$

When $p \in P_M \cap P_N$, find v, w, $v_i = p_i$, $w_i = 1-p_i$.

coeff of S in g_M.

$$\log\left(\inf_{y>0} \frac{g_M(y)}{y^p}\right) = \sum_S v_S \log\left(\frac{c_S}{v_S}\right) = \sum_{S \in B_M} v_S \log\left(\frac{1}{v_S}\right) = H(v)$$

$$H(v) \geq \sum_i v_i \log\left(\frac{1}{v_i}\right) = \sum_i p_i \log\left(\frac{1}{p_i}\right)$$

$$\log\left(\inf_{z>0} \frac{g_{N^*}(z)}{z^{1-p}}\right) \geq H(w) = H(w^*) \geq \sum_i p_i \log\left(\frac{1}{p_i}\right)$$

Implication (cont'd)

$$\left(\Phi(p) = \prod_i \frac{p_i^{p_i}(1-p_i)^{1-p_i}}{1+p_i(1-p_i)} \right)$$

$$\log |B_{nr} \cap B_n| \geq \log \Phi(p) + 2\sum_i p_i \log\left(\frac{1}{p_i}\right)$$

$$\left(\frac{p^p(1-p)^{1-p}}{1+p(1-p)} \geq p \log\left(\frac{p}{e^2}\right) \right)$$

$$\geq \sum_i p_i \log\left(\frac{p_i}{e^2}\right) + 2\sum_i p_i \log\left(\frac{1}{p_i}\right)$$

$$= \sum_i p_i \log\left(\frac{1}{p_i}\right) - 2\sum_i p_i$$

$$\geq \sum_i H(p_i) - 2r - \sum_i (1-p_i) \cdot \log \frac{1}{1-p_i}$$

$$\geq \sum_i H(p_i) - 3r.$$

Base case ($n = 1$)

- Characterize log-concavity of $g(y,z) = a + by + cz + dyz$
  - Assume $a, b, c, d \geq 0$

- Prove that

$$(\partial y + \partial z) g(y,z) \Big|_{y=z=0} \geq \frac{p^p (1-p)^{1-p}}{1 + p(1-p)} \inf_{y,z>0} \frac{g(y,z)}{y^p z^{1-p}}$$

Whiteboard

- Characterize log-concavity of $g(y,z) = a + by + cz + dyz$

$$\frac{g(y,z)\nabla^2 g(y,z) - (\nabla g(y,z))(\nabla g(y,z))^T}{g(y,z)^2}$$

$$\text{Numerator} = (a+by+cz+dyz)\begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} - \begin{bmatrix} b+dz \\ c+dy \end{bmatrix}\begin{bmatrix} b+dz & c+dy \end{bmatrix}$$

$$= \begin{bmatrix} -(b+dz)^2 & (a+by+cz+dyz)d - (b+dz)(c+dy) \\ " & -(c+dy)^2 \end{bmatrix}$$

$$\det = (b+dz)^2(c+dy)^2 - (ad-bc)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (bc)^2 \geq (ad-bc)^2 \quad \Leftrightarrow bc \geq |ad-bc| \quad \Leftrightarrow 2bc \geq ad$$

Whiteboard

- Characterize log-concavity of $g(y, z) = a + by + cz + dyz$

$g(y,z)$ log-concave $\Leftrightarrow$ $2bc - ad \geq 0$.

Whiteboard

- Prove that

$$(\partial y + \partial z) g(y,z) \Big|_{y=z=0} \geq \frac{p^p(1-p)^{1-p}}{1+p(1-p)} \inf_{y,z>0} \frac{g(y,z)}{y^p z^{1-p}}$$

$g(y,z) = a + by + cz + dyz$

$\text{LHS} = b + c$

Look at $\inf_{y,z>0} \frac{g(y,z)}{y^p z^{1-p}}$

- Special case 1: $p \in \{0,1\}$

Say $p=0$, $\inf_{y,z>0} \frac{a+by+cz+dyz}{z} \leq c$ $\quad y \to 0$, $z \to +\infty$

- Special case 2: $d=0$.

$\hookrightarrow bc = 0$, $b=0$, let $y \to +\infty$, $\inf = 0$.

$\hookrightarrow bc > 0$, by $2bc \geq ad$, scale d

Whiteboard

- Prove that

$$(\partial y + \partial z) g(y,z) \Big|_{y=z=0} \geq \frac{p^p (1-p)^{1-p}}{1 + p(1-p)} \inf_{y,z>0} \frac{g(y,z)}{y^p z^{1-p}}$$

$2bc = ad$, Scale so that $d = 1$

$$\inf_{y,z>0} \frac{2bc + by + cz + yz}{y^p z^{1-p}} \geq \inf_{y,z>0} \frac{bc + (c+y)(b+z)}{y^p z^{1-p}}.$$

$$\rightarrow y = c \cdot \frac{p}{1-p}, \quad z = b \cdot \frac{1-p}{p}.$$

$$\left(y = c \cdot \left[\frac{p}{1-p} + \sqrt{1 + \left(\frac{p}{1-p}\right)^2} - 1 \right], \; z = b \left[\frac{1-p}{p} + \sqrt{1 + \left(\frac{1-p}{p}\right)^2} - 1 \right] \right.$$

$p = \frac{1}{2}$, 5 vs $(2 + 2\sqrt{2})$.

Whiteboard

- Prove that

$$(\partial y + \partial z) g(y,z)\Big|_{y=z=0} \geq \frac{p^p (1-p)^{1-p}}{1 + p(1-p)} \inf_{y,z>0} \frac{g(y,z)}{y^p z^{1-p}}$$

Plug $y = c\,\frac{p}{1-p}$, $z = b\,\frac{1-p}{p}$.

$$\text{RHS} \leq \frac{\cancel{p^p (1-p)^{1-p}}}{1+p(1-p)} \cdot \frac{bc\left[1 + \frac{1}{1-p}\cdot\frac{1}{p}\right]}{c^p b^{1-p} \cancel{\left(\frac{p}{1-p}\right)^p} \cdot \cancel{\left(\frac{1-p}{p}\right)^{1-p}}}$$

$$= b^p c^{1-p} \cdot \frac{\cancel{1+p(1-p)}}{p(1-p)} \cdot (1-p)^p \cdot p^{1-p} \cdot \frac{1}{\cancel{1+p(1-p)}}$$

$$= \left(\frac{b}{p}\right)^p \cdot \left(\frac{c}{1-p}\right)^{1-p}$$

$$\leq b + c = \text{LHS}.$$

End of Part III